

e) $\int_2^e \sin(x) \cos(x) dx$

NOTA $\int e^g g' dx$

$$\begin{aligned} &= \int_2^e 3 \cos^2(x) \sin(x) \cos(x) dx = \int_2^e \frac{(-6)}{(-6)} \sin(x) \cos(x) dx \\ &\quad = -\frac{1}{6} \int_2^e 3 \cos^2(x) (-6) \sin(x) \cos(x) dx \\ &\quad = -\frac{1}{6} e^{3 \cos^2(x)} + C, C \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} g(x) &= 3 \cos^2(x) \\ g'(x) &= 3 \times 2 \cos(x)(-\sin(x)) \\ &= (-6) \cos(x) \sin(x) \end{aligned}$$

f) $\int \frac{e^{2x}}{1 + e^{4x}} dx = \int \frac{e^{2x}}{1 + (e^{2x})^2} dx$

$$= \int \frac{1}{2} \frac{2e^{2x}}{1 + (e^{2x})^2} dx$$

$$= \frac{1}{2} \int \frac{2e^{2x}}{1 + (e^{2x})^2} dx = \frac{1}{2} \arctan(e^{2x}) + C, C \in \mathbb{R}$$

CA
 $\int \frac{g'}{1+g^2} = \arctan(g)$
 $g = e^{2x}$
 $g' = 2e^{2x}$

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Exercício

Determina a primitiva F pl a função

$$f(x) = \frac{2}{x} + \frac{3}{x^2} \text{ tal que } \underbrace{F(-1) = 1}_{C=?}$$

$$\begin{aligned} F(x) &= \int f(x) dx \\ &= \int \left(\frac{2}{x} + \frac{3}{x^2} \right) dx \end{aligned}$$

$$\begin{aligned}
 F(x) &= \int \frac{2}{x} dx + \int \frac{3}{x^2} dx \\
 &= 2 \int \frac{1}{x} dx + 3 \int \left(\frac{1}{x^2} \right) dx \\
 &= 2 \int \frac{1}{x} dx + 3 \int x^{-2} dx \\
 &= 2 \ln|x| + 3 \frac{x^{-2+1}}{-2+1} + C \\
 &= 2 \ln|x| + 3 \frac{x^{-1}}{-1} + C \\
 &= 2 \ln|x| - \frac{3}{x} + C, \quad C \in \mathbb{R}
 \end{aligned}$$

Uma vez que $F(-1) = 1$, vem

$$F(x) = 2 \ln|x| - \frac{3}{x} + C$$

$$\begin{aligned}
 \underbrace{F(-1)}_1 &= 2 \ln|-1| - \frac{3}{(-1)} + C \\
 1 &= 2 \ln(1) + 3 + C
 \end{aligned}$$

$$C = -2$$

Então, a primitiva de f é dada por

$$F(x) = 2 \ln|x| - \frac{3}{x} - 2$$

Ex:

Sabendo que a função f satisfaz a igualdade

$$\int f(x) dx = \sin(x) - x \cos(x) - \frac{1}{2}x^2 + C, C \in \mathbb{R},$$

determine $f\left(\frac{\pi}{4}\right)$

$$F(x) = \sin(x) - x \cos(x) - \frac{1}{2}x^2 + C, C \in \mathbb{R}$$

é a primitiva de f .

$$F'(x) = f(x)$$

$$\left(\frac{1}{2}x^2\right)' = \frac{2}{2}x = x$$

$$F'(x) = \cancel{\cos(x)} - \cancel{\cos(x)} + x \sin(x) - x \\ = x(\sin(x) - 1)$$

$$F'\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \left(\sin\left(\frac{\pi}{4}\right) - 1 \right) \\ = \frac{\pi}{4} \left(\frac{\sqrt{2}}{2} - 1 \right) \\ = \frac{\pi}{4} \left(\frac{\sqrt{2} - 2}{2} \right) = \frac{\pi}{8}(\sqrt{2} - 2)$$

$$\text{Então, } f\left(\frac{\pi}{4}\right) = \frac{\pi}{8}(\sqrt{2} - 2)$$

→ Primitivação por partes

$$(fg)' = f'g + \boxed{g'f}$$

$$\Rightarrow g'f = (fg)' - f'g$$

Aplicando primitivas a ambos os membros, vem 21

$$\int g'f \, dx = fg - \int f'g \, dx$$

e portanto, se fixarmos $h(x) = g'(x)f(x)$

$$\int h(x) \, dx = \int g'(x)f(x) \, dx = fg - \int g f' \, dx$$

fórmula de primitivas por partes

Ideia: Dada a função h , temos de fazer uma escolha adequada das funções g' e f para forma a que $h = g'f$.

Regra q̄ poderia ser útil para escolha de $\boxed{g'}$

- 1º Exponenciais
- 2º Trigonométricas
- 3º algébricas ($1, x^2, x^{-2}$)
- 4º Inversas Trigonométricas
- 5º logarítmicas

Exemplo:

$$\begin{aligned} \int x^3 e^{x^2} \, dx &= \int x^2 x e^{x^2} \, dx = \int x^2 x e^{x^2} \, dx \\ &= \frac{1}{2} \int x^2 \underbrace{2x}_{\downarrow} e^{x^2} \, dx \end{aligned}$$

$$\begin{aligned} \frac{\text{ca}}{g'} \underbrace{x^2 e^{x^2}}_{f} &\Rightarrow \begin{cases} g = e^{x^2} \\ f' = 2x \end{cases} & & = \frac{1}{2} \left(x^2 e^{x^2} - \int 2x e^{x^2} \, dx \right) \\ & & & = \frac{1}{2} \left(x^2 e^{x^2} - e^{x^2} \right) + C \end{aligned}$$

$$= \frac{1}{2} e^{x^2} (x^2 - 1) + C, C \in \mathbb{R}$$

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Exemplo :

$$\int \ln(x) dx = \int 1 \times \underbrace{\ln(x)}_{g}, dx = x \ln(x) - \int x \frac{1}{x} dx$$

\downarrow \downarrow
g' f

DA

$$\begin{aligned} g' &= 1 \\ f \ln(x) &\Rightarrow \begin{cases} g = x \\ f' = \frac{1}{x} \end{cases} \end{aligned}$$

\downarrow
pela fórmula de
primitivas por partes

$$\begin{aligned} &= x \ln(x) - \int 1 dx \\ &= x \ln(x) - x + C \\ &\Rightarrow x(\ln(x) - 1) + C, C \in \mathbb{R} \end{aligned}$$

Exemplo :

$$\int x \arctg(x) dx \sim ?$$

\downarrow \downarrow
g' f

$$\begin{aligned} g' &= x \\ f &= \arctg(x) \end{aligned} \Rightarrow \begin{cases} g = \int g' dx = \int x dx = \frac{x^2}{2} \\ f' = \frac{1}{1+x^2} \end{cases}$$

$$\int x \arctg(x) dx = \arctg(x) \frac{x^2}{2} - \int \frac{1}{1+x^2} \frac{x^2}{2} dx$$

\downarrow
pela primitivas
por partes

$$\int g' f - f g - \int f' g$$

$$= \operatorname{arctg}(x) \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

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$$\frac{\cancel{x^2}}{1+x^2} = \frac{(x^2+1)-1}{(x^2+1)} = \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} = 1 - \frac{1}{x^2+1}$$



$$= \operatorname{arctg}(x) \frac{x^2}{2} - \frac{1}{2} \left(1 - \frac{1}{x^2+1} \right) dx$$

$$= \operatorname{arctg}(x) \frac{x^2}{2} - \frac{1}{2} \left(\int 1 dx - \int \frac{1}{x^2+1} dx \right)$$

$$= \operatorname{arctg}(x) \frac{x^2}{2} - \frac{1}{2} \left(x - \operatorname{arctg}(x) \right) + C$$

$$= \operatorname{arctg}(x) \frac{x^2}{2} - \frac{1}{2}x + \frac{1}{2} \operatorname{arctg}(x) + C, C \in \mathbb{R}$$

Exemplo:

$$\int (x+1) \sin(x) dx = ?$$

$$\begin{cases} g' = \sin(x) \\ f = x+1 \end{cases} \Rightarrow \begin{cases} g = -\cos(x) \\ f' = 1 \end{cases}$$

$$\begin{aligned} \int (x+1) \sin(x) dx &= (x+1)(-\cos(x)) - \int 1 \times (\cos(x)) dx \\ &= -(x+1)\cos(x) + \int \cos(x) dx \\ &= -(x+1)\cos(x) + \sin(x) + C, C \in \mathbb{R} \end{aligned}$$

Exemplo:

$$\int \ln(x^2+2) dx = \int 1 \times \underbrace{\ln(x^2+2)}_{\downarrow g} dx$$

\downarrow
g

$$\begin{aligned} g' &= 1 \\ f &= \ln(x^2+2) \end{aligned} \Rightarrow \int \cancel{g} = \infty$$

$$\int \frac{2x}{x^2+2} dx$$

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$$= x \ln(x^2 + 2) - \int x \frac{2x}{x^2 + 2} dx$$

$$= x \ln(x^2 + 2) - \int 2 \frac{x^2}{x^2 + 2} dx$$

$$= x \ln(x^2 + 2) - 2 \int \left\{ \frac{x^2}{x^2 + 2} \right\} dx$$

CA

$$\frac{x^2}{x^2 + 2} = \frac{(x^2 + 2) - 2}{(x^2 + 2)} = \frac{x^2 + 2}{x^2 + 2} - \frac{2}{x^2 + 2} = 1 - \frac{2}{x^2 + 2}$$

$$\Rightarrow = x \ln(x^2 + 2) - 2 \int \left(1 - \frac{2}{x^2 + 2} \right) dx$$

$$= x \ln(x^2 + 2) - 2 \left(\int 1 dx - \int \frac{2}{x^2 + 2} dx \right)$$

$$= x \ln(x^2 + 2) - 2 \int 1 dx + 2 \int \frac{2}{x^2 + 2} dx$$

$$= x \ln(x^2 + 2) - 2 \int 1 dx + 2 \int \frac{1}{\frac{x^2}{2} + 1} dx$$

$$= x \ln(x^2 + 2) - 2 \int 1 dx + 2 \int \frac{1}{\left(\frac{x^2}{2}\right)^2 + 1} dx$$

$$= x \ln(x^2 + 2) - 2 \int 1 dx + 2 \int \sqrt{2} \frac{\frac{1}{\sqrt{2}}}{\left(\frac{x^2}{2}\right)^2 + 1} dx$$

CA

$$\left(\frac{x}{\sqrt{2}} \right)' = \frac{1}{\sqrt{2}} x' = \frac{1}{\sqrt{2}} (x)' < \frac{1}{\sqrt{2}}$$

$$\Rightarrow = x \ln(x^2 + 2) - 2 \int 1 dx + 2\sqrt{2} \int \frac{\frac{1}{\sqrt{2}}}{\left(\frac{x^2}{2}\right)^2 + 1} dx$$

$$= x \ln(x^2 + 2) - 2x + 2\sqrt{2} \arctg \left(\frac{x}{\sqrt{2}} \right) + C, C \in \mathbb{R}$$

Exercício

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Cálcula os seguintes integrais indefinidos:

a) $\int e^{-3x} (2x+3) dx$

b) $\int \ln^2(x) dx$

c) $\int \sin(\ln(x)) dx$

d) $\int \arcsin(x) dx$

e) $\int \operatorname{arctg}\left(\frac{1}{x}\right) dx$

f) $\int \sqrt{x} \ln(x) dx$

g) $\int x \cdot 3^x dx$

i) $\int e^{2x} \sin(x) dx$

h) $\int \frac{x^2}{(1+x^2)^2} dx$

j) $\int \frac{x+2}{3} \cos(5x) dx$

g) $g' = e^{-3x}$ $\Rightarrow \begin{cases} g = \frac{e^{-3x}}{-3} \\ f = 2x+3 \end{cases}$
CA:
 $2 \times 1 = 2$

$$\begin{aligned} \int e^{-3x} (2x+3) dx &= (2x+3) \left(\frac{e^{-3x}}{-3} \right) - \int 2 \frac{e^{-3x}}{-3} dx \\ &= -\frac{(2x+3)}{3} e^{-3x} + \frac{2}{3} \int e^{-3x} dx \\ &= -\frac{(2x+3)}{3} e^{-3x} + \frac{2}{3} \frac{e^{-3x}}{-3} + C \end{aligned}$$

$$= -\frac{(2x+3)}{3} e^{-3x} - \frac{2}{9} e^{-3x} + C, C \in \mathbb{R}$$

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b) $\int \ln^2(x) dx = x \ln^2(x) - \int x \cdot 2 \ln(x) \frac{1}{x} dx$
 $= x \ln^2(x) - \int 2 \ln(x) dx$

$$\begin{cases} g' = 1 \\ f = \ln^2(x) \end{cases} \Rightarrow \begin{cases} g = x \\ f' = 2 \ln(x) \frac{1}{x} \end{cases}$$

$$= x \ln^2(x) - 2 \ln(x) dx$$

aplicando novamente a
fórmula de primitivas por
partes:

$$\begin{cases} g' = 1 \\ f = \ln(x) \end{cases} \Rightarrow \begin{cases} g = x \\ f' = \frac{1}{x} \end{cases}$$

$$\begin{aligned} &= x \ln^2(x) - 2 \left(x \ln(x) - \int \frac{1}{x} x dx \right)' \\ &= x \ln^2(x) - 2 \left(x \ln(x) - \int 1 dx \right) \\ &= x \ln^2(x) - 2x \ln(x) + 2x + C, C \in \mathbb{R} \end{aligned}$$