Problem Set IV Solution

Prepared by the Teaching Assistants

November 2013

1. Question 1

   a)

   The solution of this exercise is more mathematically involving than what was required. For the shorter version, follow through the solution provided in the practical classes.

   This exercise is a modified version of the two period model introduced in exercise 4 of Problemset 2. As before the individual chooses his consumption level between period 1 and period 2. The tax level is equal to 0 at the first period. As said the savings are subsided in such a way that the interest rate increases to $(1 + r + a)$. This subsidy (of amount a), is paid at the second period through taxation.

   For each period the individual faces one budget constraint:

   For period 1: $c_1 = y_1 - s$, which means that consumption at period 1 equals the income at that period less the savings. Note that the savings can be positive or negative.

   For period 2: $c_2 = y_2 - t_2 + (1+r+a)s$, meaning the consumption at period 2 equals the disposable income at that period plus the savings from the period before, considering the new interest rate (1+r+a).

   Picking the budget constraint for period 2 and isolating the variable s we get

   $$s = \frac{c_2}{1+r+a} + \frac{y_2 - t_2}{1+r+a}$$

   using this expression for s and substituting in the Budget constraint for the first period we get:

   $$c_1 + \frac{c_2}{1+r+a} = y_1 + \frac{y_2 - t_2}{1+r+a}$$

   From where we can take the relation between c_1 and c_2 given by:

   $$c_2 = -c_1(1 + r + a) + (y_1 - t_1)(1 + r + a) + (y_2 - t_2)$$

   In order to find the optimal levels of consumption and savings we should perform the maximization of the utility function subject to the inter temporal budget constraint:

   $$c_1 + \frac{c_2}{1+r+a} = y_1 + \frac{y_2}{1+r+a}.$$

In this way the optimization problem will be given by:

$$\begin{aligned}
& \max_{c_1,c_2} \{\log c_1 + \beta \log c_2\} \\
& s.a \\
c_1 + \frac{c_2}{1+r+a} \quad = \quad & y_1 + \frac{y_2 - t_2}{1+r+a}
\end{aligned}$$

Taking the first order conditions we get

$$\begin{cases} \dfrac{\partial L}{\partial c_1} = \dfrac{1}{c_1} - \lambda = 0 \\ \\ \dfrac{\partial L}{\partial c_2} = \dfrac{\beta}{c_2} - \dfrac{\lambda}{1+r+a} = 0 \\ \\ \dfrac{\partial L}{\partial \lambda} = y_1 + \frac{y_2-t_2}{1+r+a} - c_1 - \frac{c_2}{1+r+a} = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{c_1} = \lambda \\ \\ \dfrac{\beta}{c_2} = \dfrac{\lambda}{1+r+a} \\ \\ c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2-t_2}{1+r+a} \end{cases} . \tag{1}$$

Dividing the first two conditions we obtain,

$$\begin{cases} \frac{c_2}{\beta c_1} = 1+r+a \\ \\ c_1 + \frac{c_2}{1+r+a} = y_1 + \frac{y_2-t_2}{1+r+a} \end{cases} \Leftrightarrow \begin{cases} c_2 = (1+r)\,\beta c_1 \\ \\ c_1 + \frac{c_2}{1+r+a} = y_1 + \frac{y_2-t_2}{1+r+a} \end{cases} . \tag{2}$$

Solving for the optimal c_1 and c_2,

$$\begin{cases} c_1^* = \frac{1}{1+\beta}\left(y_1 + \frac{y_2-t_2}{1+r+a}\right) \\ \\ c_2^* = \frac{\beta}{1+\beta}\left[(1+r+a)\,y_1 + y_2 - t_2\right] \end{cases} \tag{3}$$

By the maximization routine just showed we arrived to the optimal levels of consumption at both period dependent on the income at period 1 and period 2 and on the interest rate.

To get to the optimal level of saving we must recall the budget constraint for period 1:

$$s = y_1 - c_1$$

Substituting for the optimal c_1 found above we get the optimal saving level:

$$\begin{aligned}
s^* \quad &= \quad y_1 - \frac{1}{1+\beta}\left(y_1 + \frac{y_2 - t_2}{1+r+a}\right) \Leftrightarrow \\
\Leftrightarrow \quad & s^* = \frac{\beta}{1+\beta} y_1 - \frac{y_2 - t_2}{(1+\beta)(1+r+a)} = \frac{\beta y_1 - \frac{y_2-t_2}{1+r+a}}{1+\beta}
\end{aligned}$$

We have arrived so to the optimal consumption (at the two periods) and optimal saving considering the framework given.

b)

Let us now turn to the problem of the government. At the first period it has no revenues and and no expenditures. On the second period the government as to pay the total subsidy transferred to the consumers $a * s$, meaning the subsidy per unit of savings (a) times the savings. These expenditures must be totally financed by taxes at period two: $t_2 = as$.

Considering that $y_2 = 0$ let us pick the and substitute for this new constraint:

$$c_1^* = \frac{y_1}{1+\beta} - \frac{t_2}{(1+\beta)(1+r+a)}$$

$$c_2^* = \frac{\beta(1+r+a)y_1}{1+\beta} - \frac{\beta t_2}{(1+r+a)(1+\beta)}$$

$$s^* = \frac{\beta y_1}{1+\beta} + \frac{t_2}{(1+\beta)(1+r+a)}$$

Graphically this corresponds to the following graphical representation:

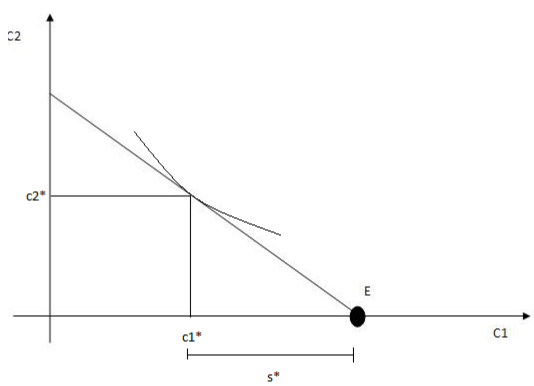

Figure 1: Initial Equilibrium

Since the individual has no income at the following period his endowment point corresponds to the point E stated in the figure. In this case the representative consumer is always a lender since he desires to save in order to consume in the next period

Given $t_2 = as$ it will be found now the level of taxes at period 2:

$$t_2 = a\frac{\beta y_1}{1+\beta} + \frac{t_2}{(1+\beta)(1+r+a)}$$

$$\frac{[(1+\beta)(1+r+a) - a]t_2}{(1+\beta)(1+r+a)} = \frac{\beta y_1 a}{(1+\beta)}$$

$$t_2 = \frac{\beta y_1 a(1+r+a)}{[(1+r+a) - a]}$$

It was found the level of taxes at period 2 we can now come back to the optimal expressions for the consumption and savings and substitute for the t_2 just found:

$$c_1^* = \frac{y_1}{1+\beta} - \frac{\beta y_1 a}{(1+\beta)[(1+\beta)(1+r+a)-a]}$$

$$c_2^* = \frac{\beta(1+r+a)y_1}{1+\beta} - \frac{\beta^2 y_1 a}{(1+\beta)[(1+\beta)(1+r+a)-a]}$$

$$s^* = \frac{\beta y_1}{1+\beta} + \frac{\beta y_1 a}{(1+\beta)[(1+\beta)(1+r+a)-a]}$$

We know that the utility of the agent is dependent on the consumption in the two periods, then we will evaluate the change in the utility after the introduction of the subsidy. Remembering the uitility of the agent:

$$U(c_1; c_2) = log c_1 + \beta \log(c_2)$$

considering the optimal relation between c_1 and c_2 derived in the maximization routine:

$$c_2 = (1+r+a)\beta c_1$$

substituting in the utility function we have:

$$U(c_1; c_2) = log(c_1) + \beta \log[(1+r+a)\beta c_1]$$

$$U(c_1) = (1+\beta)log(c_1) + \beta \log[(1+r+a)\beta]$$

Substitute for the optimal c_1.

$$U(c_1) = (1+\beta)log[\frac{y_1}{1+\beta} - \frac{\beta y_1 a}{(1+\beta)(1+r+a)-a}] - (1+\beta)log(1+\beta) + \beta \log[(1+r+a)\beta]$$

$$U(c_1) = (1+\beta)log(y_1) + (1+\beta)log[\frac{1}{1+\beta} - \frac{\beta a}{(1+\beta)[(1+\beta)(1+r+a)-a]}] - (1+\beta)log(1+\beta) + \beta \log[(1+r+a)\beta$$

$$U(c_1) = (1+\beta)log(y_1) + (1+\beta)log[1 - \frac{\beta a}{[(1+\beta)(1+r+a)-a]}] - (1+\beta)log(1+\beta) + \beta \log[(1+r+a)\beta]$$

Taking now the derivative of the above expression in relation to a:

$$\frac{\partial\left(U(c_1)\right)}{\partial a}=\beta\left(\frac{1}{1+r+a}-\frac{1+\beta}{(1+\beta)(1+r)+\beta a}\right)$$

$$\frac{\partial\left(U(c_1)\right)}{\partial a}=-\frac{a\beta}{(1+r+a)[(1+\beta)(1+r)+\beta a}<0$$

From the expression above we can denote that the introduction of the subsidy bring the individual to a lower level of utility.

This happened since the creation in the subsidy and in the taxes lead to the following graphical representation:

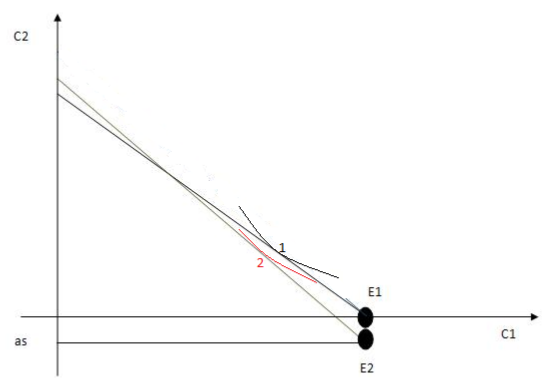

Figure 2: Final Equilibrium

As can be observed, if by one side it is true that the slope of the budget constraint moved due to the increase in the interest rate, through the introduction of the subsidy, by the other side we have a downwards movement, since the subsidy must be paid as taxes in t_2. Then once again an introduction of a distortion in the economy is responsible by the break in the equilibrium and consequently by the deterioration of the individual's welfare

c)

In the model the savings made by the agents corresponds to the exports of this economy. To measure the impact of the subsidy in the exports we take the derivative of the optimal savings found above in relation to a:

$$\frac{\partial\left(s^*\right)}{\partial a}=\frac{\beta y_1(1+\beta)(1+r)}{(1+\beta)[(1+r+a)-a]^2}>0$$

Then the increase in the subsidy leads to an increase in exports, or an increase in savings.

At this point and given the results already found in the previous two questions we can state some conclusions from this exercise:

1. By one side it is true that the subsidy leads to an increase in savings and consequent increase in exports

2. Although this subsidy as to be paid in form of taxes the period after, at t_2, which makes that in the aggregate effect the on individuals will be worse than without the subsidy.

d)

$1+r = \frac{1}{\beta}$ and $\beta = 1$ means that the interest rate will be equal to 0. Picking the optimal conditions for consumption and savings found in question b), but substituting for the new conditions stated:

For $a > 0$ we have:

$$c_1^* = \frac{y_1}{2} - \frac{y_1 a}{4+2a}$$

$$c_2^* = \frac{(1+a)y_1}{2} - \frac{y_1 a}{4+2a}$$

$$s^* = \frac{y_1}{2} + \frac{y_1 a}{4+2a}$$

Once again recalling that the utility function written in function of c_1 we get:

$$U(c_1) = (1+\beta)log(c_1) + \beta \log[(1+r+a)\beta]$$

$$U(c_1) = (1+\beta)log(\frac{y_1}{2} - \frac{y_1 a}{4+2a}) - (2)log(2) + \log[(1+a)]$$

As before taking the derivative of the utility in relation to a we get:

$$\frac{\partial\left(U(c_1)\right)}{\partial a} = -\frac{a}{(1+a)(2+a)} < 0$$

About savings the derivative is given as:

$$\frac{\partial\left(s^*\right))}{\partial a} = \frac{y_1}{(2+a)^2} > 0$$

As in the case before the introduction of taxes decreases the level of welfare, although it leads to an increase in the savings at the first, savings that will serve to pay the taxes at the second period.

Then in terms if welfare it is better $a = 0$ than $a > 0$

2. Question 2

(a) Assume there are N consumers. In a previous problem set we have seen that an intertemporal maximisation problem as the one given, yields the following optimal savings:

$$s_i^* = \frac{1}{1+\beta}\left[\beta y_1 - \frac{y_2}{1+r}\right]$$

In the absence of government, aggregate savings will be aggregate private savings, that is:

$$s^* = S^P = \sum_{i=1}^{N} s_i^* = \frac{N}{1+\beta}\left[\beta y_1 - \frac{y_2}{1+r}\right].$$

The savings curve can be obtained by rewriting the optimal savings in order to $1+r$ - for simplicity consider $s^* = \dfrac{S^*}{N}$:

$$1+r = \frac{y_2}{\beta y_1 - (1+\beta)\, s^*}$$

Regarding its slope, we can find whether it is positive or not by taking the derivative:

$$\frac{\partial\,(1+r)}{\partial s^*} = \frac{y_2\,(1+\beta)}{\left[\beta y_1 - (1+\beta)\, s^*\right]^2} > 0.$$

Given that both $y_2 > 0$, savings and the interest rate are positively related.

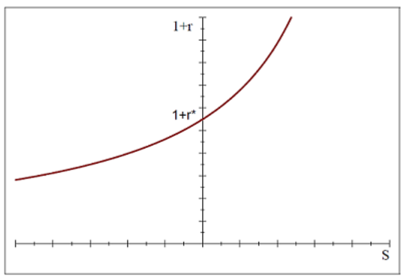

Figure 3: Savings Curve

Because there is no investment nor government, the equilibrium interest rate is found by equating aggregate savings to zero $S^* = 0$.

(b) Equating aggregate savings to zero $S^* = 0$ gives:

$$1+r^* = \frac{y_2}{\beta y_1}$$

(c) We are told that the interest rate is $4\% \Rightarrow r = 0.04$ and that the output growth rate is $2\% \Rightarrow y_2/y_1 = 1.02$ (since everyone has the same endowment, the ratio between current and future individual income must be equal to output growth). Assuming the credit market is in equilibrium, we know that

$$1+r = \frac{y_2}{\beta y_1}.$$

Replacing by the values given, we obtain

$$\begin{aligned} 1.04 &= \frac{1.02}{\beta} \\ \beta &= \frac{1.02}{1.04} = 0.98. \end{aligned}$$

Thus, the value of β compatible with these observations is 0.98.

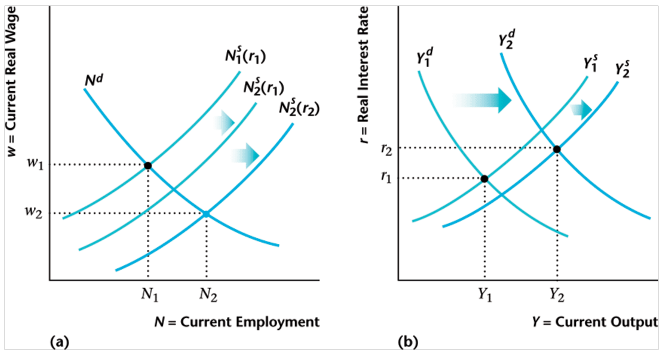

Figure 4: Increase in G Temporarily

3. Question 3

   (a)

   An increase in G temporarily leads the output demand curve Y^D to shift to the right by ΔG - given that the indirect effects in consumption via the present value of wealth we ($\uparrow Y^D$ and $\uparrow PV_T$) cancel one another. Henceforth, the output demand multiplier is one.
   On the supply side, and increase in G leads to a decrease in the present value of wealth we. Income effects tells us that leisure will decrease and so the labour supply N^S increases. This causes the output supply curve Y^S to increase as well (for any given level of the interest rate).
   Notice, however, that a temporary change in income does not change lifetime income by much. Meaning that the resulting changes in N^S and Y^S are small. And so, because $|\Delta Y^S| < |\Delta Y^D|$ the interest rate is higher in the new equilibrium. To support it, the N^S curve will shift to the right again - since an increase in r increases the relative cost of current leisure in terms of future leisure.
   The corresponding results are: $\uparrow N; \downarrow w; \uparrow Y; \uparrow r$. However, Y increases by less than G - because r increased, we have the crowding-out effect in both C and I.
   For a more complete justification check pages 426-431 of the book (5^{th} edition).

(b)

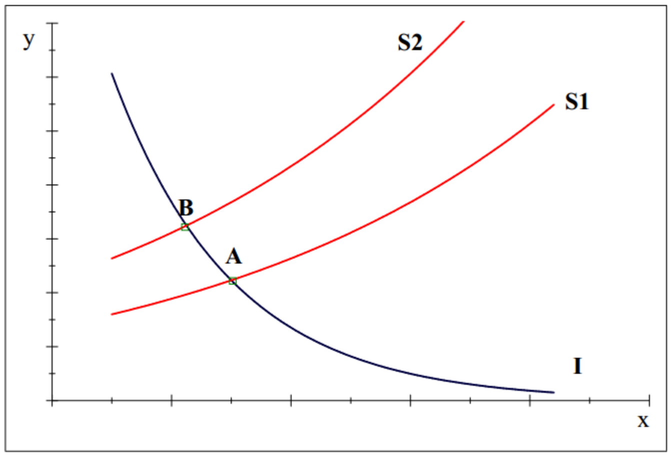

Figure 5: Savings and Investment Curves

If current taxes increase, the lifetime wealth we of the consumer decreases. However, due to the consumption smoothing behaviour, consumption will decrease by less than income. In this way, private savings will have to decrease. The dispersion of income between today and the future encourages agents to borrow money.
Since public savings $S^G = 0$, aggregate savings will be $S = S^P$. Given this, the aggregate savings curve will contract increasing the interest rate and decreasing aggregate investment as well.

(c) If the government decides to finance the increase in government expenditure through bonds, we know that future taxes will increase - the government has to satisfy its intertemporal budget constraint. Since future taxes increase, the future income of the consumer will decrease. As such and to smooth consumption, the consumer will have to increase savings in order to "transfer" income from the present to the future.
All things considered, public savings will decrease ($S^G = -B$), since the government issues debt ($B > 0$) while private savings increase since the consumer wants to smooth consumption across the present period and the future.

(d) No, either way of financing the increase in public expenditure will lead to the same results in a). From the Ricardian Equivalence we know that for a given present value of government expenditure, the distribution of taxes to finance it between today and in the future is not relevant ($PV_G = G + \frac{G'}{1+r} = T + \frac{T'}{1+r} = PV_T$). The consumer anticipates that if the government finances its current expenditures through debt, this will imply higher future taxes. Thus leaving him with the same lifetime wealth as the case where the government decided to finance his expenditures by means of an increase in current taxes.

4. Question 4.

(a) To do the analysis we need to focus o the Labor Market (figure 4 and 6) and the Goods Market (figure 5). Let the timing be indexed by the sub index, beginning in 0, followed by 1, and so on and so forth. An increase in productivity will raise the Marginal Productivity of Labor, with the consequent increase in demand of Labor and Output. That will induce a drop in the interest rate and then a decrease in the supply of labor.

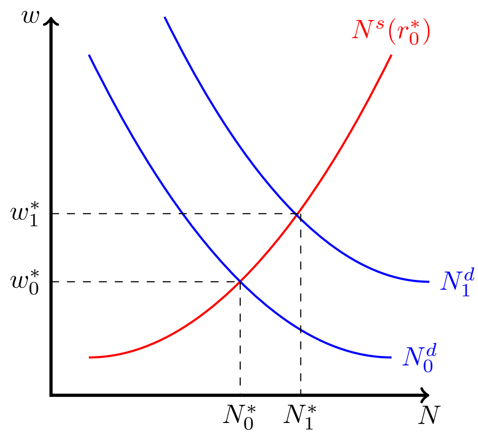

Figure 6: In the Labor Market, first move

Then in the Goods Market,

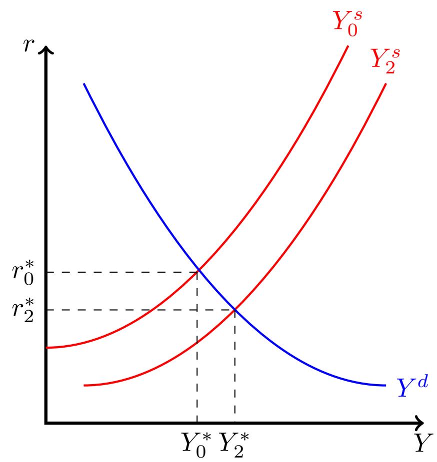

Figure 7: In the Goods Market, second move

Then finally in the Labor market again, because of the movement of the interest rate:

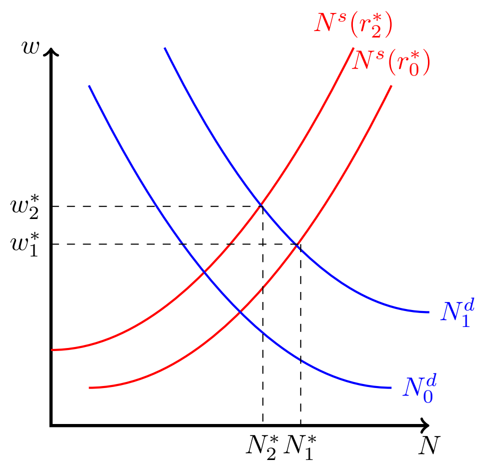

Figure 8: In the Labor Market, third move

(b) Now we know we are going to be more productive and receiving a higher wage in the second period. Then we increase the demand today(figure 7) and acquire loans that will be payed with that higher income later. There is no change in the Supply because the productivity today has not changed at all. Then the interest rate increases (because of the higher demand for loans) and that causes an expansion in the labor supply(figure 8). Finally we have a higher interest rate, higher Labor and lower wages.

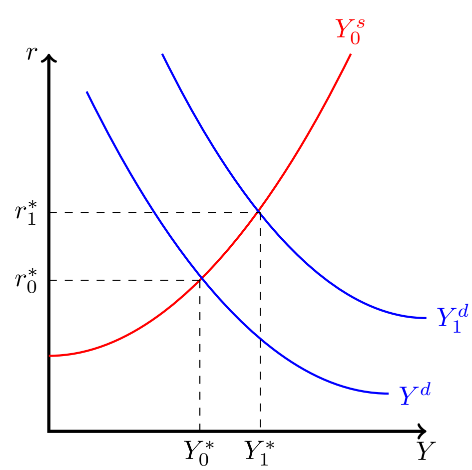

Figure 9: In the Goods Market, first move

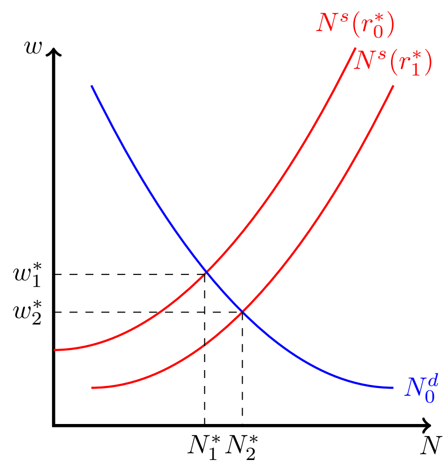

Figure 10: In the Labor Market, second move

(c) As depicted in (a) an increase in current productivity will raise the GDP and because of the increase in savings the interest rate will drop. Now first, the increase in GDP will cause a higher demand for money, because more transaction are going to take place in the economy. On the other hand a drop in the interest rate will increase as well the demand of money, because the opportunity cost of holding money is lower (or the price of money decrease, nevertheless this would be when talking about the nominal interest rate). So unambiguously the Demand for money will increase. In the assumption of fixed constant money supply the price level would decrease(figure 9).

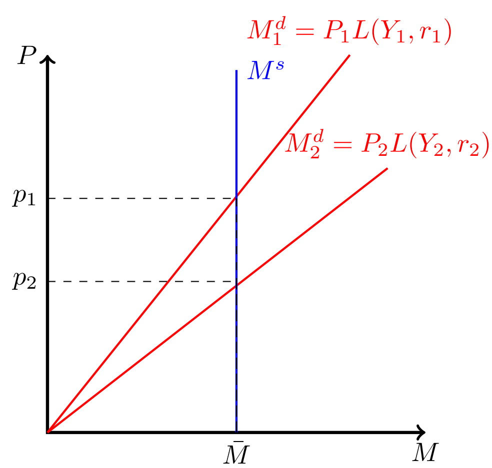

Figure 11: Money Market