Problem Set III Solution

TA Solution

October 2013

1. Question 1

   (a) The answer to this first question mimics the answer that was given to question 5 a) of PS 2. We are under a 1 period closed economy model with three agents: a representative consumer, a representative firm and the government. We know that the government has an exogenous level of expenditures given by G. These expenditures are financed by lump-sum taxes (T) paid by consumers. Since this model just accounts for one period the public budget must be always balanced, $G = T$. In order to arrive to the competitive equilibrium for this economy, we must state the problems for the consumer and the firm.
   Consumers solve:

$$\begin{aligned}&\max_{c,l} u(c,l)\\&st \quad c = (h-l)w + \pi - T\end{aligned}$$

Deriving our Lagrangian to solve this maximization problem:

$$L = \log c + \log l + \lambda\left[(h-l)w + \pi - T - c\right]. \tag{1}$$

Taking FOC of our problem:

$$\begin{cases}\dfrac{\partial L}{\partial l} = \dfrac{1}{l} - \lambda w = 0\\[2ex] \dfrac{\partial L}{\partial c} = \dfrac{1}{c} - \lambda = 0\\[2ex] \dfrac{\partial L}{\partial \lambda} = (h-l)w + \pi - T - c = 0\end{cases} \Leftrightarrow \begin{cases}\dfrac{1}{l} = \lambda w\\[2ex] \dfrac{1}{c} = \lambda\\[2ex] (h-l)w + \pi - T = c\end{cases}. \tag{2}$$

From the two first FOC we get:

$$\frac{\partial u/\partial l}{\partial u/\partial c} = w$$

Which is the everyday Marginal Rate of Substitution between consumption and leisure, equalizing the slope of the budget constraint, in this case the ratio of prices w.
Now for the firms, we know:

$$\max_{N} \pi = zF(K,N) - wN$$

From which we can take FOC, $z\frac{\partial F}{\partial N} - w = 0$, implying that $z\frac{\partial F}{\partial N} = w$. This optimal condition is equivalent to state that: $MP_{Nd} = w$

From the maximization problem of the consumer it is possible to derive the labour supply, while from the maximization problem of the firm we can arrive to our labour demand. These two sides compose the labour market, and the market is cleared when the labour supply equals the labour demand. When that occurs we arrive to our competitive equilibrium, where is achieved the optimal condition stated by:

$$MRS = MP_N = MRT = W$$

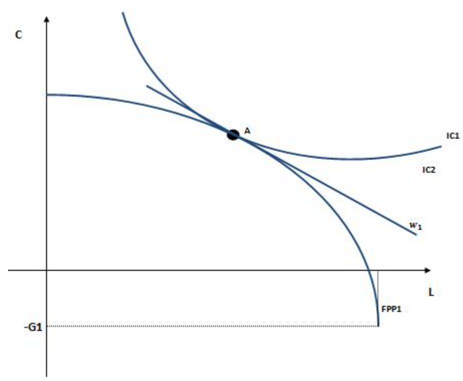


Figure 1: Competitive Equilibrium

Here we put together the PPF, whose shape derives from the production function, and the indifference curve for the consumer. Through the initial optimal point (A) passes a tangent line with slope w. This optimality representation is the graphical mirror of the optimal condition stated before.

This is our baseline graphic. Every shock in the model departs from this equilibrium picture. In the question it is suggested an increase in the Government expenditures, which causes the following graphical changes:

As can be observed we have a movement from the optimal point (A) to the optimal point (B) where we have a drop in the level of consumption and leisure and a fall in the wage: $w_2 < w_1$

Let us see step by step what happens in this economy:

1. The exogenous level of expenditures (G) increases. Since we are in a one period model and the public expenditures must equal the taxes, so the lump-sump taxes (T) will increase as well

2. We want to observe how the representative agent reacts after the increase in taxes, meaning how the consumption and leisure levels will change. To do that let us recall from PS1, the conditions for optimal consumption and optimal leisure found in question 5 c):

$$l = \frac{w+\pi-T}{2w}$$

$$c = \frac{w+\pi-T}{2}$$

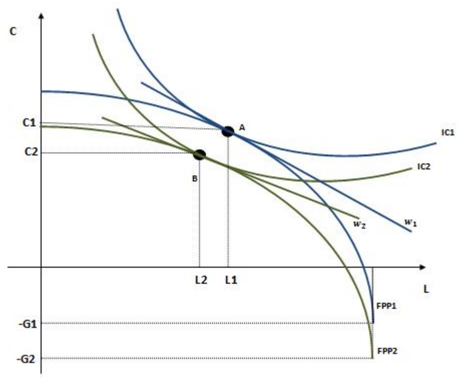


Figure 2: Competitive Equilibrium after an increase in G

From these two optimal conditions we can observe that if lump-sum taxes (T) increase the consumption and leisure levels decrease.
3.If the level of leisure decreases, that is equivalent to state that the representative agent is available to supply more hours of labour in the economy. This is reflected in the expansion of the labour supply that can be observed below:

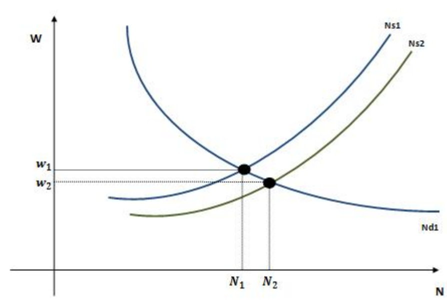


Figure 3: Labour market after the increase in G

4. The expansion in the labour supply will cause the decrease of the wage from w_1 to $w_(2)$.
5. Recalling that the Output in this economy is given as:

$$Y = C + G$$

From the output equation above we can conclude that there is two effects working over the output level: by one side the government expenditures are increasing, by the other side (and due to the increase in G)the consumption is falling. So what is the total aggregate effect on output (Y)? To conclude about it let us remember that the Y can be given as:

$$Y = zF(K, N)$$

From the properties of the production function we know that:

$$\frac{\partial Y}{\partial N} > 0$$

This implies that if we increase the amount of hours worked in the economy, that will necessarily increases the output. As it is stated in point 3, the hours work increase, so that implies that Y increases.

Therefore in the aggregate effect we know that:

$$\triangle G > 0; \triangle Y > 0; \triangle C < 0$$

In Figure 3 we can sum up all the facts stated above: The level of consumption is lower, as well the hours of leisure. The fact that the wage decreases makes the slope of the line tangent to the new optimal point (B) be flatter that the tangent line on the previous optimal point (A).

(b) The shock considered now affects not the Government expenditures, but the total factor productivity of the economy, given by z. After the shock in productivity the production function shifts as is presented in the graph below

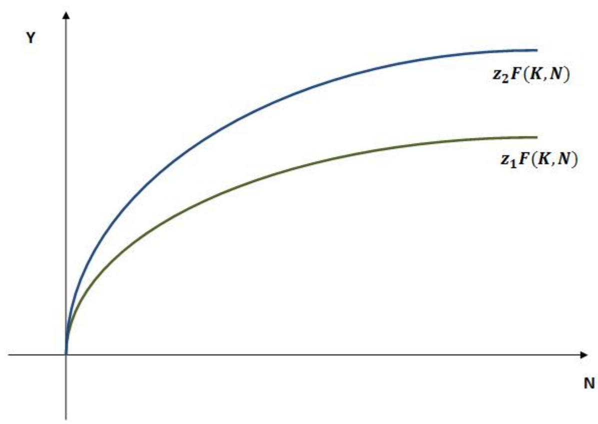


Figure 4: Change in z

In this graphical representation, we fixed the capital, K, and we can observe that after the increase in the factor z, for the same level of labour N, we can achieve an higher level of output, Y.

Considering our competitive equilibrium framework, the effect of the increase in z leads to:

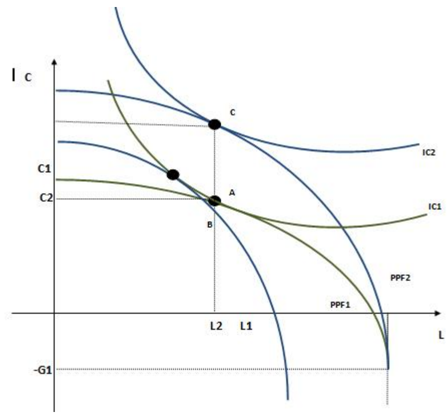


Figure 5: Change in z, competitive equilibrium

Let us see this change step by step:

1. The shock in z changes the slope of the PPF with the movement from the PPF1 to PPF2. This increase in productivity was responsible for the expansion of the possible possibilities in the economy, as can be denoted in the graph

2. We now want to measure the evolution from point A to point C. This movement can be divided into two moments: a substitution effect and an income effect.

The substitution effect: As usual in a substitution effect we must consider the new slope, in this case of the PPF, passing at a tangent point in the old indifference curve, IC1. This tangency condition is given at point B, where the level of leisure is lower and consumption is higher.

The income effect is denoted by the movement from point B to point C. In this movement we have a parallel shift of the the PPF, and its slope is not changed. This effect is responsible for the increase in leisure and consumption. Once again we cannot guarantee what will be the final point in terms of leisure, since it depends on consumer's preferences and on the size of the substitution and income effect. For simplification in the graph shown we end up at the same level of leisure.

3. We must complete this analysis, referring the impact over the wage from the increase in z. We will divide the evolution of the wage in the change from point A to point B and then from point B to point C.

From point A to point B the slope of the PPF changed, since we assume the increase in z. In point B we are over a point in the indifference curve for which the MRS increases, which imply , by our optimality condition that our wage will increase as well.

Then from point B to point C we are over the new level of technology, z_2, however we have a shift in the PPF by a fixed amount. Since the quantity of leisure increases, we will be in a point where the hours of work decrease and the Marginal productivity of labour is greater, which makes the wage increase once again.

If we want to observe this change in the wage through the change in the slope of the tangency lines in the optimal points, we can say that the line that passes through point B is steeper that the one that passes through point A, and the slope of the line that passes through C will be steeper than in point B. The fact that the tangency line has a steeper slope automatically means that the equilibrium real wage in our economy must be greater.

As we can denote changes in the total factor productivity necessarily means that the representative consumers are driven to an higher indifference curve, which implies an higher level

of utility.

(c) Let us recall the correlations table that was used in the answer of exercise 3 of PS2:

Table 1: Correlations

	real private consumption	Real Investment	employment	labor productivity ≈ average wage
GDP	0,75	0,20	0,086	0,25

We have seen that private consumption, government expenditures, employment and labour productivity are procyclical variables in relation to the GDP.

When we have performed the increase in the Government expenditures, in question a) we have seen that the output and employment increases, although the private consumption and the labour productivity (proxy for the real wage) decreased. These results of the model do not confirm what is observed in the real data, as it can be seen by the table above. In this way, changes in G do not seem to be a good source to explain the generation of business cycles. On the other way we have seen that when we have a positive shock in z, the output increases, as well private consumption and the real wage. We are not sure about the effect in employment, since we do not know the magnitude of the substitution and the income effect. However changes in z seem to fit much better what is observed in real data, meaning that the fluctuations in the productivity factor (z) are a likely candidate to explain the business cycles in the economy.

2. Question 2:

(a) We need to state the maximization problem of the agent:

$$\max_{c,l} u(c,l) = \log(c) + \log(l)$$

$$s.t \quad c = (h-l)z$$

And as usual, take first order conditions:

$$\frac{\partial u}{\partial l} = 0 \quad \Rightarrow \quad \frac{1}{c_1}\frac{\partial c}{\partial l} + \frac{1}{l} = 0$$

Now from the budget constraint $\frac{\partial c}{\partial l} = -z$ so replacing we obtain

$$\frac{z}{c} = \frac{1}{l}$$

Which can be rearranged to obtain

$$\frac{c}{l} = z$$

With this we can replace into the budget constraint and finally have our consumption and leisure. Rewrite the optimality condition as: $c = zl$, and replace c in the BC.

$$\begin{aligned} zl &= (h-l)z \\ zl &= hz - zl \\ 2zl &= hz \\ l &= \frac{h}{2} \end{aligned}$$

And therefore consumption is $c = \frac{hz}{2}$.

(b) To have the correct graph it turns out to be crucial having the previous part done. Actually we found leisure to be constant. That means that for any change we impose, leisure must be equal at the end. In particular in this case, facing an increase in z we would require the Substitution Effect (SE) and Income Effect (IE) to cancel each other.
Note also that the slope of the budget constraint is precisely z in the same way the wage was in our previous examples in old problem sets. So the effect would be similar:

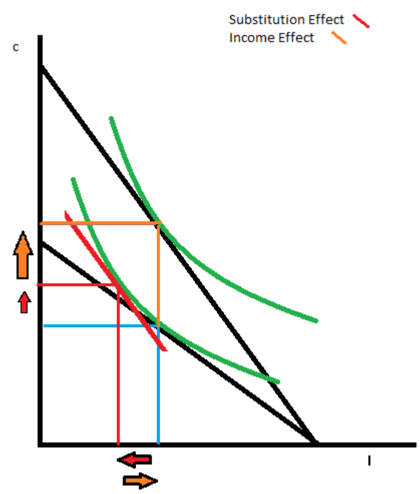


Figure 6: Total Effect of an increase in z

Note that we go back to the same leisure level we had at the beginning. This is because the leisure is independent of the shock. In red the first change, only moving the slope along the original Utility level to have the SE and then the parallel movement to have te IE. Consumption increases and leisure remains equal.

(c) In this part we are going to exploit the fact: $\frac{\partial \log(x_t)}{\partial t} = \gamma_x$ where γ_x is the growth rate of x. In this case we know one simple relation between consumption and z, recall, the optimal amount of consumption was $c = \frac{hz}{2}$. Let's work from there:

$$\begin{aligned} c &= \frac{hz}{2} \quad \text{/taking logs} \\ \log(c) &= \log(h) + \log(z) - \log(2) \quad \text{/taking derivative with respect to time} \\ \frac{\partial \log(c)}{\partial t} &= \frac{\partial \log(h)}{\partial t} + \frac{\partial \log(z)}{\partial t} - \frac{\partial \log(2)}{\partial t} \quad \text{/Impose that } \frac{\partial \log(x)}{\partial t} = \gamma_x \\ \gamma_c &= 0 + \gamma_z - 0 = g \end{aligned}$$

Note that we used the fact that $\log(h)$ and $\log(2)$ are constants, so taking the derivative makes them zero. So the growth rate of consumption is zero. Leisure has growth rate of 0, because is a constant, as found in (a).

(d) Our problem would change to:

$$\max_{c,l} \log(c) + \log(l)$$

$$st \quad c = (h-l)w$$

Which we know leads to the optimality condition $\frac{c}{l} = w$ from last problem set. On the other hand, the firm maximizes:

$$\max_{N} zN - wN$$

Which leads to the F.0.C. $z - w = 0$ implying $z = w$. So the optimality imposes $z = w$, then replacing above, we would have the same conditions we found in (a).

3. Question 3

With taxes proportional to income the budget constraint of the consumer becomes

$$c = w(1-\tau)N + \pi.$$

In terms of labour and leisure the budget constraint corresponds to

$$c = w(1-\tau)(h-l) + \pi.$$

Regarding the production function, it takes the same form as in the previous exercise

$$Y = zN.$$

In this way, in equilibrium two different conditions will hold. The firm will set

$$MP_N = w \Leftrightarrow z = w,$$

while the consumer will set

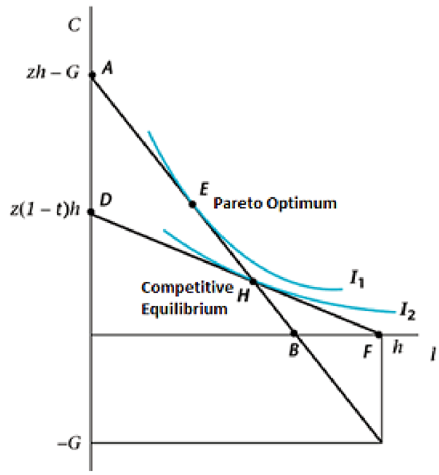
$$MRS_{l,c} = w(1-\tau).$$

Note that the consumer is no longer setting the marginal rate of substitution equal to the real wage, w, because for $\tau > 0$, $w(1-\tau) < w$. This implies that the relative price of leisure in terms of consumption is now lower since the consumer receives less money for each additional hour he works. Despite this fact, the firm still pays a wage w for each hour worked, as the firmś optimal condition shows. This way, in the competitive equilibrium the representative consumer and representative firm will not face the same price and so they will have different optimizing conditions ($MRS_{l,c}$ will be different from MP_N) - causing a distortion in the economy. Consequently, unlike the standard case in which we have lump sum taxes, the equilibrium will not be Pareto Optimum.

What are the practical implications of this distortion? Since the relative price of leisure is now lower, the consumer will choose more leisure and less consumption than what he would if he was optimizing for the real wage without taxes, w. Furthermore, since leisure will be higher, labour supply, N^s, will necessarily be lower. As such output, Y, will decrease as well.

The following graph depicts this problem:

An increase in τ would further distort our equilibrium in the direction described above.



4. Question 4.

 (a) In this exercise we leave the previous trade-off for the consumer under a 1 period economy model. In that model the individual chose between consumption and leisure. For this case we have a two period economy, and so the consumer's choice will be between consuming today and consuming tomorrow.

 Fro each period the individual faces one budget constraint:

 For period 1: $c_1 = y_1 - s$, which means that consumption and period 1 equals the income at that period less the savings. Note that the savings can be positive or negative

 For period 2: $c_2 = y_2 + (1 + r)s$, meaning the consumption at period 2 equals the income at that period plus the returns from the savings made before (if $s > 0$ at period 1) or the payment of what was borrowed (if $s < 0$ in period 1).

 Picking the budget constraint for period 2 and isolating the variable s we get

$$s = \frac{c_2}{1+r} + \frac{y_2}{1+r}$$

 using this expression for s and substituting in the Budget constraint for the first period we get:

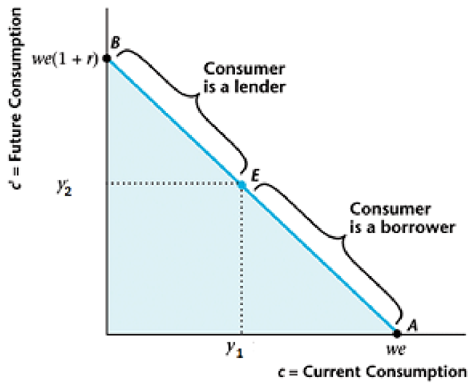
$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

 From where we can take the relation between c_1 and c_2 given by:

$$c_2 = c_1(1 + r) + y_1(1 + r) + y_2$$

 This relation can be depicted in the graph presented below:

 E corresponds to what is called as the endowment point, being the point over the line where the agent just consumes the income he has in each period: $c_1 = y_1$ and $c_2 = y_2$. This implies that the level of savings must be equal to zero: $s = 0$.



(b) In order to find the optimal levels of consumption and savings we should perform the maximization of the utility function subject to the intertemporal budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}.$$

In this way the optimization problem will be given by:

$$\begin{aligned} & \max_{c_1, c_2} \{\log c_1 + \beta \log c_2\} \\ & s.a \\ c_1 + \frac{c_2}{1+r} &= y_1 + \frac{y_2}{1+r} \end{aligned}$$

Taking the first order conditions we get

$$\begin{cases} \dfrac{\partial L}{\partial c_1} = \dfrac{1}{c_1} - \lambda = 0 \\ \dfrac{\partial L}{\partial c_2} = \dfrac{\beta}{c_2} - \dfrac{\lambda}{1+r} = 0 \\ \dfrac{\partial L}{\partial \lambda} = y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{c_1} = \lambda \\ \dfrac{\beta}{c_2} = \dfrac{\lambda}{1+r} \\ c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \end{cases}. \tag{3}$$

Dividing the first two conditions we obtain,

$$\begin{cases} \frac{c_2}{\beta c_1} = 1 + r \\ c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \end{cases} \Leftrightarrow \begin{cases} c_2 = (1+r)\,\beta c_1 \\ c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \end{cases}. \tag{4}$$

Solving for the optimal c_1 and c_2,

$$\begin{cases} c_1^* = \frac{1}{1+\beta}\left(y_1 + \frac{y_2}{1+r}\right) \\ c_2^* = \frac{\beta}{1+\beta}\left[(1+r)\,y_1 + y_2\right] \end{cases} \tag{5}$$

By the maximization routine just showed we arrived to the optimal levels of consumption at both period dependent on the income at period 1 and period 2 and on the interest rate.

To get to the optimal level of saving we must recall the budget constraint for period 1:

$$s = y_1 - c_1$$

Substituting for the optimal c_1 found above we get the optimal saving level:

$$\begin{aligned} s^* &= y_1 - \frac{1}{1+\beta}\left(y_1 + \frac{y_2}{1+r}\right) \Leftrightarrow \\ \Leftrightarrow \quad s^* &= \frac{\beta}{1+\beta}y_1 - \frac{y_2}{(1+\beta)(1+r)} = \frac{\beta y_1 - \frac{y_2}{1+r}}{1+\beta} \end{aligned}$$

(c) Considering that the income at the first period is equal to zero, the first mathematical analysis that we must do is in the optimal levels of c_1, c_2 and s substitute for $y_1 = 0$:

$$\begin{aligned} s^* &= \frac{-\frac{y_2}{1+r}}{1+\beta} \\ c_1^* &= \frac{1}{1+\beta}\left(\frac{y_2}{1+r}\right) \\ c_2^* &= \frac{\beta}{1+\beta}y_2. \end{aligned}$$

Economically this results can be read in the following way: faced with no income in the first period, the individual decides to borrow all the money that he needs to consume at period 1. That is why the savings level is negative and $s = -c_1$. Then in the next period, the individual receives y_2, and with that he consumes and pays the bonds sold in the period before. Since he cannot save at the first period, the consumption at c_2 is necessary lower when $y_1 = 0$.

(d) In this exercise it is proposed that we study the effects of an increase in the present income, y_1. This impact must be measured taking the derivatives in relation to y_1 in the optimal levels previously calculated for c_1, c_2 and s:

$$\frac{\partial s^*}{\partial y_1} = \frac{\beta}{(1+\beta)} > 0$$

$$\frac{\partial c_1^*}{\partial y_1} = \frac{1}{1+\beta}$$

$$\frac{\partial c_2^*}{\partial y_1} = \frac{\beta}{1+\beta}(1+r) > 0$$

We can observe that all the three variables have positive derivatives in relation to y_1, meaning all the variables increase if the income at period 1 rises. Intuitively this means that if income rise at period 1, the individual uses this new income to consume today, but he will increase the level of savings as well, savings that he will later use to consume more at the next period at c_2. We have here a mirror of the smoothing consumption theory, given that facing a higher income the individual prefers not use everything today for consumption, but saving a part of it, making that the consumption will increase in both periods. The consumption smoothing theory was already reflected in the preferences of the consumer. Since the utility function represents convex preferences, which stand for a consumer that values diversity. For this

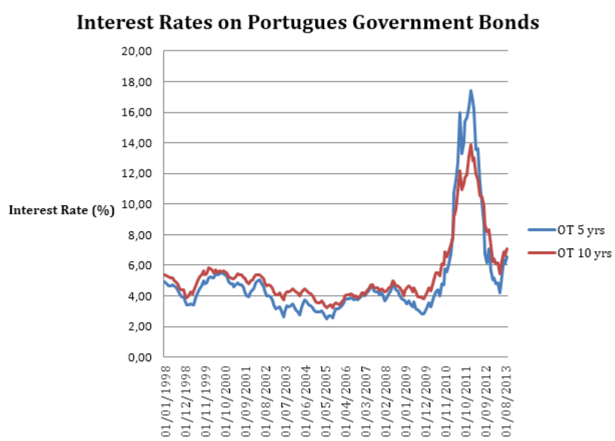
particular case, this diversity imply that the agent likes to balance the consumption between the two periods.

In order to ask to the second part of this exercise it must be understood that the individual passes from a borrower to a lender when passes the endowment point, and at this turn over point we know by definition that $s = 0$. Picking so our expression for the optimal savings we get:

$$s^* = \frac{\beta y_1 - \frac{y_2}{1+r}}{1+\beta} = 0 \Leftrightarrow y_1 = \frac{y_2}{\beta(1+r)}.$$

So when y_1 equals $\frac{y_2}{\beta(1+r)}$, we are passing through the endowment point, which is equivalent to passing from a borrower to a lender.

(e) Question 5.



The graph depicts the evolution of the interest rate on Portuguese Government Bonds with maturities of 5 and 10 years. This is monthly data extracted from *BPstat* and reports to a period ranging from January 1998 to September 2013. Government Bonds are debt securities issued by the Government to support its expenses. The Government can issue bonds with different maturities. When the repayment is due, it shall have to pay the face-value of the bond plus interest. The terms on which a government can sell bonds are determined by how credit-worthy the markets perceive our country to be. The popular Rating Agencies' assessments have a direct impact on this market perception, hence the interest rates attached to our debt-issuing. The graph shows that interest rates on bonds of 5 and 10-year maturities move closely together across the whole period. Up until 2010, interest rates were fairly low, moving roughly between 3 and 6%. Higher maturity bonds were relatively higher, which is the typical scenario: investors usually perceive bonds with higher maturities to be riskier. From 2010 onwards, the situation completely changed. 5 and 10-year bonds rose sharply due to an increase in the perceived risk of Portuguese bonds triggered by the financial crisis (by January 2012, our Government was being asked $17,42\%$ of interest on 5-year bonds). Also, from March 2011 to June 2012 lower maturity bonds held a higher interest than 10-year ones. That may be explained by the investors' belief that the Portuguese economy would recover in the long-run, thus perceiving longer maturity bonds to be less risky than short-term ones. The situation transmuted again throughout 2012. By May 2013, interest on 5-year bonds was down to $4,21\%$ (recall the surge of good news upholding GDP recovery in the first trimester of the present year). In the past few months it has increased back to the 7% level though.