Problem Set II

October 2013

1. Question 1.

   (a) The production function of this economy is given by:

$$Y = zK^a N^{1-a}$$

In general, parameter a is determined through the first order conditions derived for the firm's maximization problem, assuming the markets for inputs operate in perfect competition. The firm's maximization problem is given by:

$$\max_{N,K} Y - rK - wN.$$

note: here we are assuming that capital K is not fixed and that the cost of each unit is r , the interest rate.

Taking the first order conditions:

$$\begin{cases} \dfrac{\partial Y}{\partial K} - r = 0 \\ \dfrac{\partial Y}{\partial N} - w = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{\partial Y}{\partial K} = r \\ \dfrac{\partial Y}{\partial N} = w \end{cases}.$$

These imply that, at the optimum, the firms will choose quantities of the two inputs so that their marginal productivity is equal to their marginal cost or market price. Since $Y = zK^a N^{1-a}$, we know that:

$$\begin{aligned} \frac{\partial Y}{\partial K} &= zaK^{a-1}N^{1-a} \\ &= a\frac{Y}{K} \end{aligned}$$

$$\begin{aligned} \frac{\partial Y}{\partial N} &= z(1-a)K^a N^{-a} \\ &= (1-a)\frac{Y}{N}. \end{aligned}$$

Plug these back into the first order conditions and obtain the following:

$$\begin{cases} a = \dfrac{rK}{Y} \\ 1 - a = \dfrac{wN}{Y} \end{cases}.$$

As such, a is considered to be the % contribution of capital to output, while $(1-a)$ is the % contribution of labour to output. Given that it is easier to obtain estimations for wN than for rK, the second condition is generally used to estimate the value of parameter a.

Parameter a can then be estimated once we obtain time series for w, N and Y. Since the variability of wN/Y is usually low, it is reasonable to take its average throughout the period under analysis, thus obtaining an unique value for a.

Empirically, for developed economies, like the U.S., $a \approx 1/3$. In other words, roughly 33% of total production is due to capital contribution while the rest amounts for the labor contribution. For greater detail check pages 146-147 from the text book.

(b) This exercise could be solved in either discrete or continuous time.

In *discrete* time, the evolution of capital and labour can be described by the following equations:

$$\begin{aligned} K_{t+1} &= (1+g)\, K_t \\ N_{t+1} &= (1+n)\, N_t \end{aligned}$$

While the output Y at time t is given by the production function (assuming z to be constant - no growth rate):

$$Y_t = zK_t^a N_t^{1-a}$$

And at $t+1$ by:

$$Y_{t+1} = zK_{t+1}^a N_{t+1}^{1-a}.$$

Since we know the growth rates of capital and labor we can rewrite output in period $t+1$ as a function of K_t and N_t:

$$Y_{t+1} = z\left[(1+g)\, K_t\right]^a \left[(1+n)\, N_t\right]^{1-a}.$$

Now, recall our discussion in the previous Problem Set. We have seen that growth rates can be derived by using logs in the following way:

$$g_Y = \log\left(Y_{t+1}\right) - \log\left(Y_t\right).$$

Replacing $\log\left(Y_{t+1}\right)$ and $\log\left(Y_t\right)$ in the previous expression we get:

$$\begin{aligned} g_Y &= \log\left[z\left[(1+g)\, K_t\right]^a \left[(1+n)\, N_t\right]^{1-a}\right] - \log\left[zK_t^a N_t^{1-a}\right] \\ &= \log z + a\left[\log(1+g) + \log K_t\right] + (1-a)\left[\log(1+n) + \log N_t\right] \\ &\quad - \log z - a\log K_t - (1-a)\log N_t \\ &= a\log(1+g) + (1-a)\log(1+n) \end{aligned}$$

Recall that $\log(1+x) \simeq x$, for a small value of x, the growth rate of output is approximately:

$$ag + (1-a)\, n,$$

thus, the growth rate of output corresponds to weighted average of the growth rate of each input (weighted by the average contribution of each input to the production process).

In *continuous* time, the production function is given by:

$$Y(t) = zK(t)^a N(t)^{1-a}.$$

Applying logarithms, we obtain

$$\log Y(t) = \log z + a \log K(t) + (1-a) \log N(t).$$

Note that the growth rate of output is given by the first derivative of its logarithm in order to time.

$$g_Y = \frac{\partial \log Y(t)}{\partial t} = \frac{\frac{\partial Y(t)}{\partial t}}{Y(t)} = \frac{\dot{Y}(t)}{Y(t)} \tag{1}$$

In this way, taking the first derivative of the whole expression we get the output growth rate.

$$\begin{aligned} g_Y &= a\frac{\dot{K}(t)}{K(t)} + (1-a)\frac{\dot{N}(t)}{N(t)} \\ &= ag + (1-a)n \end{aligned}$$

We get the exact same result!

2. Question 2.

If we consider a short average between now days and 6 months ago, we see that there is a slight increase in the interest rate (or spread depend on graph), for 10 or 5 years bonds. You can have an explanation from class and look at the slides of Macroeconomics Facts.

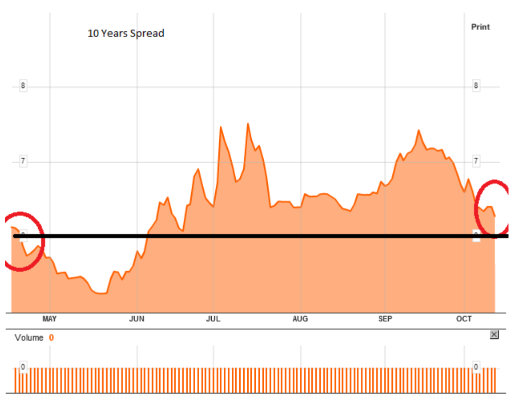

Figure 1: 10yr Portuguese Bond Spread

Nevertheless, the spreads over the sovereign bonds depends in many factors. It can depend directly on the relevant big macro variables of an economy (for example if GDP is going down, unemployment going up, you could expect difficulties for the Government to raise enough taxes to repay the debt, so risk increases, and then the spread).

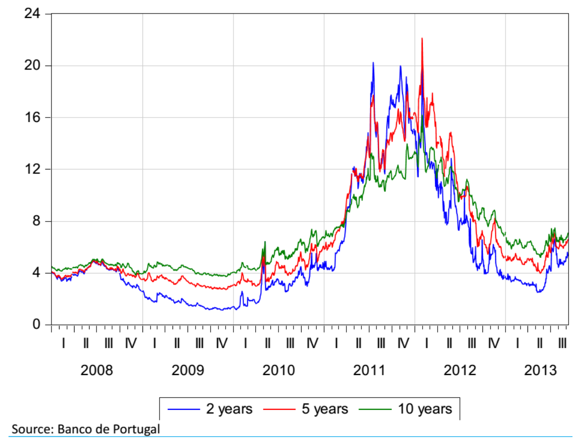

Figure 2: 3yr, 5yr, and 10yr Portuguese Bond Spread from Theoretical Lectures

Another source of increasing spreads is political instability, specially if the country cannot agree on measures to reactivate the economy to repay the debt. Moreover, even if there is consensus, if that consensus is a default, it will definitively raise the interest rates.

Another source could be region instability. For example, if another European country in troubles, is expected to need extra help, all the investors will evaluate more severely the countries who share the weak position, so the spreads would increase by contagion.

Finally (but not the last, there are many more) institutional reforms (or lack of them) imposed by third parties (IMF, EU, BCE, etc) could have a great impact. For example the positive impact in reducing the spreads when BCE's president Mario Draghi said he would do anything is needed to preserve the stability in the region tranquilized the markets decreasing the spreads.

Now, a pure explanation would be in line that people is saving more in Government bonds, so the "price" decreases, therefore the spread.

(Note: in the description above are potential hints of what could have driven the bonds to move in the way depicted in the graph, however it was not required an explanation as deep as the one exposed).

3. Question 3

   To observe if the variables are procyclical or countercyclical we took quarterly data about the GDP, Private Consumption, Public expenditure, Gross Formation of Fixed Capital and average wage for the portuguese economy between 1978 and 2012. To note that the average wage is used as a proxy for the labor productivity variable, since we assume that from the profit maximization problem of the firm we get:

$$\frac{\partial Y}{\partial N} = w$$

   The table resumes the correlations

   As we can observe all the variables are procyclical.

Table 1: Correlations

	real private consumption	Real Investment	employment	labor productivity $\approx$ average wage
GDP	0,75	0,20	0,086	0,25

4. Question 4

   Income taxes distort the labor market. Why will there be a distortion? In our model consumers are optimizing with $MRS = w(1-t)$ while firms continue to optimize with $MP_N = w$. This *tax wedge* gives rise to a mismatch between the firms' and consumers' decisions and the resulting competitive equilibrium will no longer be Pareto Optimum. The consumers will choose more leisure than would be optimal (the case where taxes are lump sum), meaning less production and less consumption. Check pages 184-187 of the book (5th edition) for further detail.

   Our model predicts that an increase in income tax t will have the same effect in the consumer as a decrease in his effective real wage $w(1-t)$ (pages 129-131). If so, labor supply will change and the - already present - distortion will be inflated. Therefore, differences in taxes may explain differences found in labour supplies across countries. Clemens' article presents evidence that more taxes encourages people to work less hours. It also presents estimations of the costs of the distortion created between the competitive equilibrium and the Pareto Optimum. Thus reinforcing our conjecture of a negative relationship between income/proportional taxes and labour supply.

5. Question 5

   (a) We are under a 1 period closed economy model with three agents: a representative consumer, a representative firm and the government. We know that the government has an exogenous level of expenditures given by G. These expenditures are financed by lump-sumps taxes (T) paid by consumers. Since this model just accounts for one period the public budget must be always balanced, $G = T$ In order to arrive to the competitive equilibrium for this economy, we must state the problems for the consumer and the firm.
   Consumers solve:

$$\max_{c,l} u(c,l)$$
$$st \quad c = (h-l)w + \pi - T$$

   Deriving our Lagrangian to solve this maximization problem:

$$L = \log c + \log l + \lambda\left[(h-l)w + \pi - T - c\right]. \tag{2}$$

   Taking FOC of our problem:

$$\begin{cases} \dfrac{\partial L}{\partial l} = \dfrac{1}{l} - \lambda w = 0 \\ \dfrac{\partial L}{\partial c} = \dfrac{1}{c} - \lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} = (h-l)w + \pi - T - c = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{l} = \lambda w \\ \dfrac{1}{c} = \lambda \\ (h-l)w + \pi - T = c \end{cases} . \tag{3}$$

From the two first FOC we get:

$$\frac{\partial u/\partial l}{\partial u/\partial c} = w$$

Which is the everyday Marginal Rate of Substitution between consumption and leisure, equalizing the slope of the budget constraint, in this case the ratio of prices w.

Now for the firms, we know:

$$\underset{N}{max} \pi = zF(K,N) - wN$$

From which we can take FOC, $z\frac{\partial F}{\partial N} - w = 0$, implying that $z\frac{\partial F}{\partial N} = w$. This optimal condition is equivalent to state that: $MP_{Nd} = w$

From the maximization problem of the consumer it is possible to derive the labour supply, while from the maximization problem of the firm we can arrive to our labour demand. These two sides compose the labour market, and the market is cleared when the labour supply equals the labour demand. When that occurs we arrive to our competitive equilibrium, where is achieved the optimal condition stated by:

$$MRS = MP_N = MRT = W$$

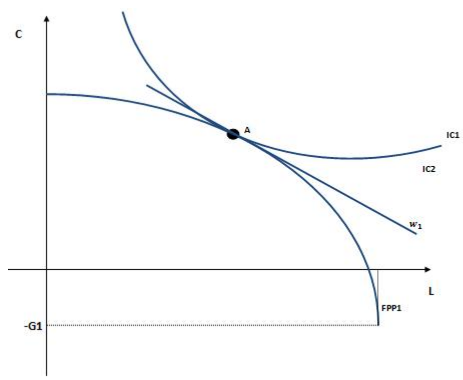

Figure 3: Competitive Equilibrium

Here we put together the PPF, whose shape derives from the production function, and the indifference curve for the consumer. Through the initial optimal point (A) passes a tangent line with slope w. This optimality representation is the graphical mirror of the optimal condition stated before.

This is our baseline graphic. Every shock in the model departs from this equilibrium picture. In the question it is suggested an increase in the Government expenditures, which causes the following graphical changes:

As can be observed we have a movement from the optimal point (A) to the optimal point (B) where we have a drop in the level of consumption and leisure and a fall in the wage: $w_2 < w_1$

Let us see step by step what happens in this economy:

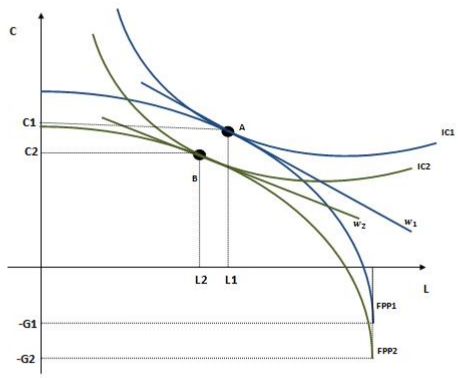

Figure 4: Competitive Equilibrium after an increase in G

1. The exogenous level of expenditures (G) increases. Since we are in a one period model and the public expenditures must equal the taxes, so the lump-sump taxes (T) will increase as well
2. We want to observe how the representative agent reacts after the increase in taxes, meaning how the consumption and leisure levels will change. To do that let us recall from PS1, the conditions for optimal consumption and optimal leisure found in question 5 c):

$$l = \frac{w+\pi-T}{2w}$$

$$c = \frac{w+\pi-T}{2}$$

From these two optimal conditions we can observe that if Lump-sump taxes (T) increase the consumption and leisure levels decrease.
3.If the level of leisure decreases, that is equivalent to state that the representative agent is available to supply more hours of labour in the economy. This is reflected in the expansion of the labour supply that can be observed below:

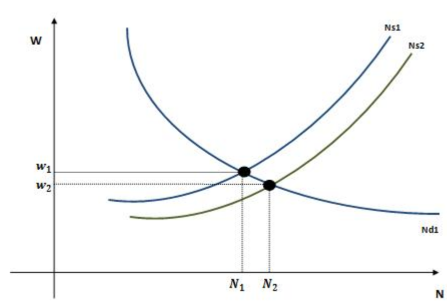

Figure 5: Labour market after the increase in G

4. The expansion in the labour supply will cause the decrease of the wage from w_1 to $w_(2)$.
5. Recalling that the Output in this economy is given as:

$$Y = C + G$$

From the output equation above we can conclude that there is two effects working over the output level: by one side the government expenditures are increasing, by the other side (and due to the increase in G)the consumption is falling. So what is the total aggregate effect on output (Y)? To conclude about it let us remember that the Y can be given as:

$$Y = zF(K, N)$$

From the properties of the production function we know that:

$$\frac{\partial Y}{\partial N} > 0$$

This implies that if we increase the amount of hours worked in the economy, that will necessarily increases the output. As it is stated in point 3, the hours work increase, so that implies that Y increases.

Therefore in the aggregate effect we know that:

$$\triangle G > 0; \triangle Y > 0; \triangle C < 0$$

In Figure 3 we can sum up all the facts stated above: The level of consumption is lower, as well the hours of leisure. The fact that the wage decreases makes the slope of the line tangent to the new optimal point (B) be flatter that the tangent line on the previous optimal point (A).

(b) From exercise 3 we can observe that Consumption, Government Expenditures and wages are procyclical variables. From the shock in Government expenditures that was performed in the previous question we know that it provokes the increase in the output, although it leads to the decrease in consumption and wages. These last two effects contradict the previous ones observed in the model. Therefore shocks in the government expenditures do not seem a suitable source to explain the generation of business cycles in the economy.

(c) In this question it is proposed to change the shape of our production function, which become a linear function, just dependent on labour:

$$Y = zN$$

The first implication from this new production function will be in the problem of the firm. Remembering that the firm is maximizing its profits we have:

$$\underset{N}{max} \pi = zN - wN$$

From this maximization problem we have that now the optimal condition for the firm is given by: $w = z$. This implies an elastic demand of labour, meaning that for every level of hours supplied by the workers the firm will always pay a wage equal to z.

More than just changing the labour market, this new production function will change the shape of our PPF which becomes linear. Repeating the shock in the Government expenditures will generate the graphs that are in Figures 6 and 7.

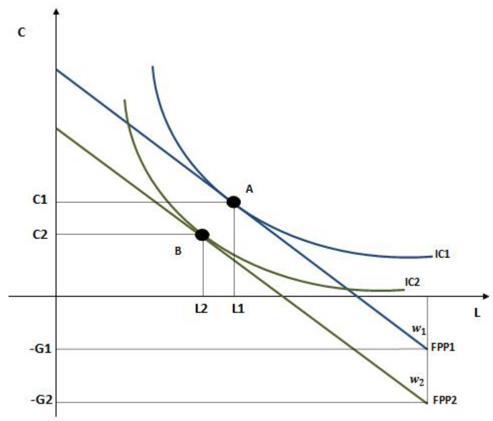

Figure 6: Increase in G with a linear PPF

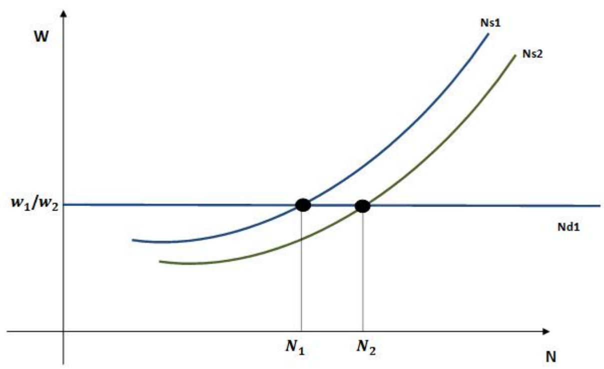

Figure 7: Labour Market after increase in G with elastic labour demand

As before the increase in the Government expenditures is responsible by the fall in the consumption and leisure, leading to an higher supply of hours in the economy. Although since the labour supply is elastic, now the wage will be stable, remaining constant from point A to point B. The constant wage after the shock is reflected in the fact that the slope of our linear PPF (z) did change from point A to point B.