Nova School of Business and Economics
Macroeconomics 1103, 2012-2013, 1st Semester
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Problem Set 2
TA Solution

1. Consumer preferences are given by,

$$U(c,l) = \log c + \log l, \tag{1}$$

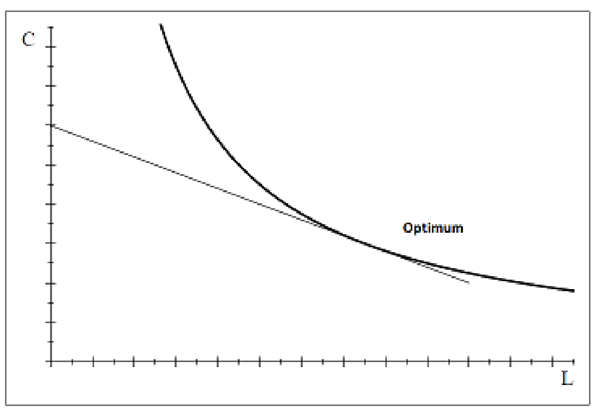
while the budget constraint is given by,

$$c \leq w(h-l) + \pi - T. \tag{2}$$

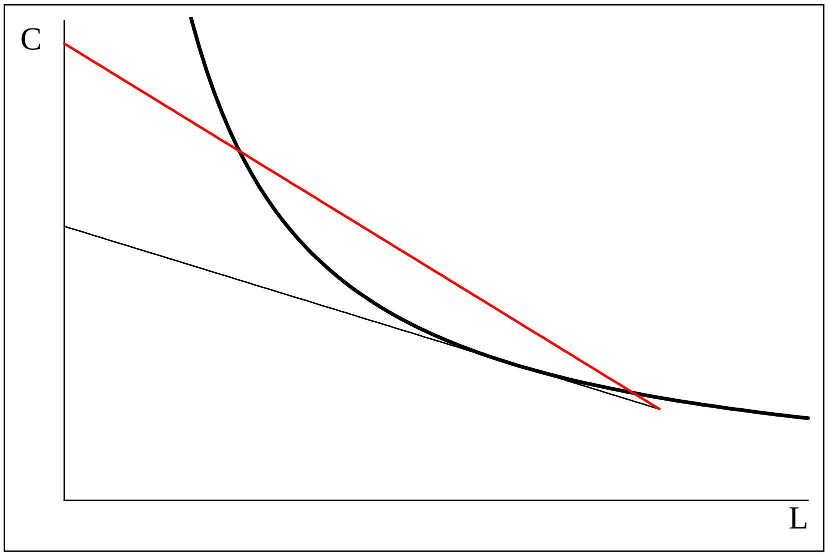
a. From the first problem set we know that the optimal choice of the representative consumer is

$$\begin{aligned} c^* &= \frac{wh+\pi-T}{2} \\ l^* &= \frac{wh+\pi-T}{2w}. \end{aligned} \tag{3}$$

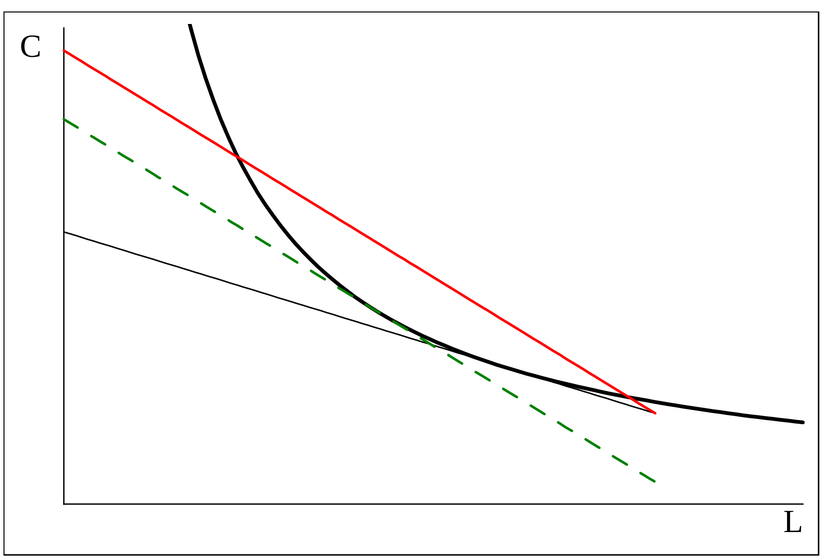
As in the previous problem set the graphical representation of the optimal choice corresponds to the tangency between the consumer's indifference curves and the budget constraint.



Let's consider an increase in the real wage from w_0 to w_1. Since the real wage is the slope of the budget constraint it will become steeper. Given this, the new budget constraint will be represented by the line in red, note that the endowment point, given by $(l,c) = (h, \pi - T)$ (the corner point), remains the same.

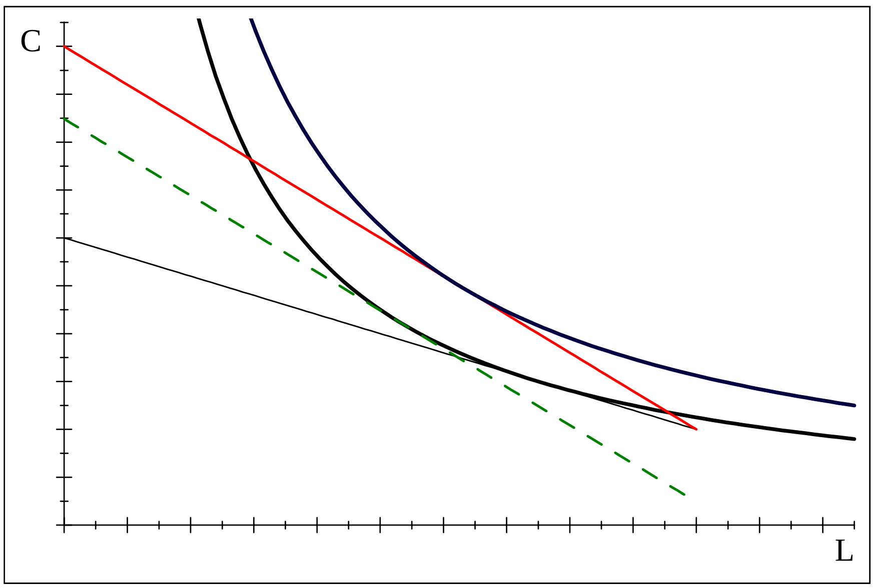


The substitution effect is the variation of the optimal choice that relates solely to the change in relative prices (this is, keeping the utility of the representative consumer fixed at its initial level). In this way, to determine the intermediate bundle of consumption that represents the substitution effect, we need to plot a fictitious budget constraint with the same slope as the final one (w_1) but tangent to the initial indifference curve. That curve corresponds to the green dashed line in the following graph:



Looking at the graph, one easly realises that, given the increase in real wage, the substitution effect has a negative impact over the choice of leisure and a positive impact over the choice of consumption. This is the case because an increase in the real wage is implictly an increase of the opportunity cost of leisure: it is now more expensive to rest!

Regarding the income effect, it is represented by the transition between the intermediate and final choice of the consumer. Note that the income effect is positive since the increase of the real wage corresponds to a raise in the value of the consumer´s time endowment (h has a higher value now). Since both consumption (c) and leisure (l) are normal goods, the amount chosen of each good will increase. The income effect is represented in the graph bellow:



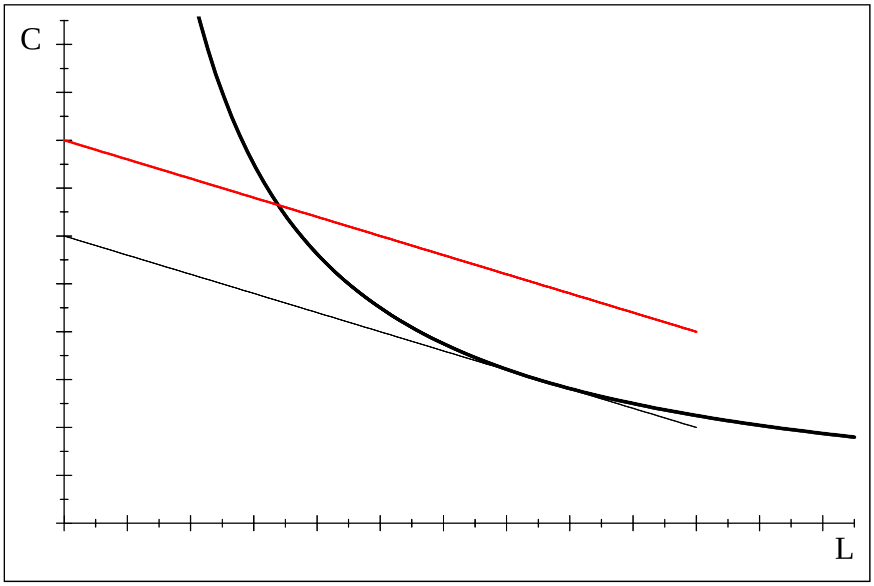
Note that, given the opposing impacts the substitution and income effect have on leisure, the net effect will be determined by the particular characteristics of each consumer (preferences, exogenous income, size of the increase in w, etc...). On the other hand, the impact on consumption (c) is clear, the consumer always chooses a higher quantity.

b. In this second case, we will have a pure income effect. Assuming that both goods are normal, this is

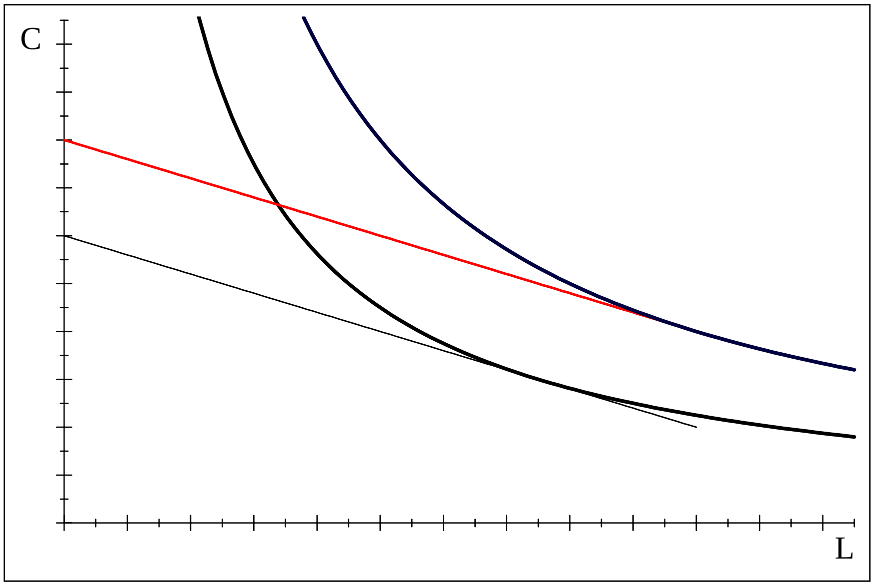
$$\frac{\partial c}{\partial M}, \frac{\partial l}{\partial M} > 0, \tag{4}$$

where M stands for the income of the consumer, we may conclude that there will be a positive variation on the optimal choice of both c and l. To begin with,

an increase in $\pi - T$ will generate a parallel expansion of the budget constraint as depicted in the following graph:



Since relative prices do not change (w remains the same), it is not necessary to analyse the substitution effect. Furthermore, since the optimal choice of leisure increases, there will be a corresponding contraction of the labor supply, for the same real wage. The final choice is represented bellow:



c. Formally, the problem is solved through a Langrange function. In this way the problem is given by

$$\max_{l,c} \quad \ln c + \ln l \tag{5}$$

$$s.a.$$

$$c = w\left(h - l\right) + \pi - T. \tag{6}$$

Writing the Lagrange function we get

$$L = \ln c + \ln l + \lambda \left[w\left(h - l\right) + \pi - T - c\right]. \tag{7}$$

The first order conditions are given by,

$$\left\{\begin{array}{l} \dfrac{\partial L}{\partial l} = \dfrac{1}{l} - \lambda w = 0 \\ \\ \dfrac{\partial L}{\partial c} = \dfrac{1}{c} - \lambda = 0 \\ \\ \dfrac{\partial L}{\partial \lambda} = w\left(h - l\right) + \pi - T - c = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \dfrac{1}{l} = \lambda w \\ \\ \dfrac{1}{c} = \lambda \\ \\ w\left(h - l\right) + \pi - T = c \end{array}\right. . \tag{8}$$

Dividing the first condition by the second we get

$$\left\{\begin{array}{l} \dfrac{c}{l} = w \\ \\ w\left(h - l\right) + \pi - T = c \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} c = wl \\ \\ w\left(h - l\right) + \pi - T = c \end{array}\right. . \tag{9}$$

Finally, solving the system for c and l we obtain the optimal choice of the consumer

$$\begin{cases} l^* = \dfrac{wh + \pi - T}{2w} \\ \\ c^* = \dfrac{wh + \pi - T}{2} \end{cases} . \tag{10}$$

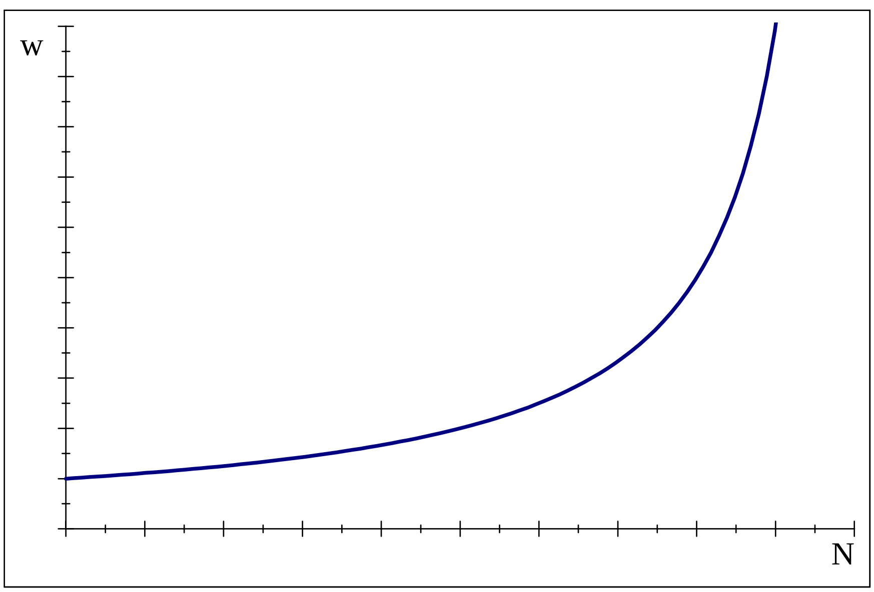
From l^* we can also obtain the optimal choice of labor which is

$$\begin{aligned} N^* &= h - l^* && (11) \\ &= h - \frac{wh + \pi - T}{2w} && (12) \\ &= \frac{wh - (\pi - T)}{2w}. && (13) \end{aligned}$$

The optimal choice of labor might also be understood as the labor supply. Solving the previous expression in order to the real wage w we get,

$$\begin{aligned} N &= \frac{wh - (\pi - T)}{2w} \Leftrightarrow 2wN = wh - (\pi - T) \Leftrightarrow && (14) \\ &\Leftrightarrow (2N - h)\, w = -(\pi - T) \Leftrightarrow w = \frac{\pi - T}{h - 2N}. && (15) \end{aligned}$$

This function is plotted in the following graph, representing the labor supply curve.



Given the preferences of the representative consumer, the labor supply curve has a positive slope, implying that the substitution effect is stronger than the income effect when the real wage w increases.

An alternative approach to determine whether the substitution effect is stronger than the income effect is to take the first derivative of the optimal choice of leisure l^* in order to the real wage w as follows,

$$\frac{\partial l^*}{\partial w} = \frac{(wh + \pi - T)' \, 2w - (2w)' \, wh + \pi - T}{4w^2} \tag{16}$$

$$= \frac{2wh - 2wh - 2(\pi - T)}{4w^2} \tag{17}$$

$$= -\frac{2(\pi - T)}{4w^2} \tag{18}$$

Assuming that $\pi - T > 0$, the sign of the derivative will be negative (given that the denominator is always positive for $w \neq 0$). This implies that the optimal choice of leisure decreases when the real wage increases, this is, the substitution effect is stronger. This is consistent with the positive slope of the labor supply curve.

For the case where $\pi - T = 0$, the derivative will be equal to zero. This implies that the substitution effect has the same size as the income effect, cancelling each other. In this case the labor supply curve would be perfectly rigid, this is a vertical line. Thus, regardless of the real wage offered, the workers always offer the same amount of labor.

2. The production function of this economy is given by

$$Y = zK^a N^{1-a}. \tag{19}$$

a. The interpretation of parameter a is obtained through the first order conditions derived from the representive firm's maximization problem, assuming the markets for inputs operate in perfect competition. The firm´s maximization problem is given by

$$\max_{N,K} Y - rK - wN. \tag{20}$$

Note: here we are assuming that capital K is not fixed and that the cost of each unit is r, the interest rate.

Taking the first order conditions

$$\begin{cases} \dfrac{\partial Y}{\partial K} - r = 0 \\ \\ \dfrac{\partial Y}{\partial N} - w = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{\partial Y}{\partial K} = r \\ \\ \dfrac{\partial Y}{\partial N} = w \end{cases}. \tag{21}$$

These two conditions tell us that, at the optimum, the firms will choose quantities of the two inputs so that their marginal productivity is equal to their

marginal cost or market price. Since $Y = zK^a N^{1-a}$, we know that

$$\begin{aligned} \frac{\partial Y}{\partial K} &= zaK^{a-1}N^{1-a} \\ &= a\frac{Y}{K}, \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial Y}{\partial N} &= z(1-a)K^a N^{-a} \\ &= (1-a)\frac{Y}{N}. \end{aligned} \tag{23}$$

Putting the previous derivations together with the first order conditions we obtain the following,

$$\begin{cases} a = \dfrac{rK}{Y} \\ \\ 1-a = \dfrac{wN}{Y} \end{cases} . \tag{24}$$

Given that it is easier to obtain estimations for wN than for rK, the second condition is generally used to estimate the value of parameter a. Intuitively, it might be understood as the percentual contribution of capital to total output.

In practice, how do we obtain an estimation of this value? Note that,

$$a = 1 - \frac{wN}{Y}.$$

Given this condition, the value of parameter a might be estimated once we obtain time series for w, N and Y. Since the variability of wN/Y is usually low, it is reasonable to take its average throughout the period under analysis, thus obtaining an unique value for a. Empirically, for developed economies, like the U.S., $a = 1/3$, in other words, roughly 33% of total production in such an economy is due to capital contribution while the rest to labor contribution. For greater detail see pages 134-135 from the text book.

b. The production function (in continuous time) is given by,

$$Y(t) = zK(t)^a N(t)^{1-a}. \tag{25}$$

Applying logarithms, we obtain

$$\log Y(t) = \log z + a \log K(t) + (1-a)\log N(t). \tag{26}$$

The growth rate of output is given by the first derivative of its logarithm in order to time,

$$g_Y = \frac{\partial \log Y(t)}{\partial t} = \frac{\frac{\partial Y(t)}{\partial t}}{Y(t)} = \frac{\dot{Y}(t)}{Y(t)} \tag{27}$$

In this way, taking the first derivative of the whole expression we get the output growth rate.

$$\begin{aligned} g_Y &= a\frac{\dot{K}(t)}{K(t)} + (1-a)\frac{\dot{N}(t)}{N(t)} && (28) \\ &= a\gamma + (1-a)\,n && (29) \end{aligned}$$

Alternatively, we may solve the exercise in discrete time. We already know that the evolution of capital and labor is given by the following equations

$$\begin{aligned} K_{t+1} &= (1+\gamma)\,K_t, && (30) \\ N_{t+1} &= (1+n)\,N_t. && (31) \end{aligned}$$

Assuming that z is constant , output in a given period t is equal to

$$Y_t = zK_t^a N_t^{1-a} \tag{32}$$

while in period $t+1$ is equal to

$$Y_{t+1} = zK_{t+1}^a N_{t+1}^{1-a}. \tag{33}$$

Since we know the growth rates of capital and labor we can rewrite output in period $t+1$ in terms of capital and labor from period t,

$$Y_{t+1} = z\left[(1+\gamma)\,K_t\right]^a \left[(1+n)\,N_t\right]^{1-a}. \tag{34}$$

From the first problem set we already know that the ouput growth rate is roughly equal to

$$g_Y = \log(Y_{t+1}) - \log(Y_t). \tag{35}$$

Replacing $\log(Y_{t+1})$ and $\log(Y_t)$ in the previous expression we get

$$\begin{aligned} g_Y &= \log\left[z\left[(1+\gamma)\,K_t\right]^a \left[(1+n)\,N_t\right]^{1-a}\right] - \log\left[zK_t^a N_t^{1-a}\right] \\ &= \log z + a\left[\log(1+\gamma) + \log K_t\right] + (1-a)\left[\log(1+n) + \log N_t\right] \\ &\quad - \log z - a\log K_t - (1-a)\log N_t \\ &= a\log(1+\gamma) + (1-a)\log(1+n) \end{aligned}$$

Recalling that $\log(1+x) \simeq x$, for a small value of x, the growth rate of output is approximately

$$a\gamma + (1-a)\,n, \tag{36}$$

thus corresponding to weighted average of the growth rate of each input (weighted by the average contribution of each input to the produtin process).

3. The text book has a detailed discussion on this matter in pages 124-125. In addition, it also relevant to see the linear production model with proportional taxes in pages 176-181.

The main issue is that, in this example, the taxes are not lump-sum, thus influencing the optimal choice of the consumer. The main implication is that the first welfare theorem will not apply any more and that the competitive equilibrium will not coincide with the Pareto Optimum. In this way, this type of taxes (proportional) will generate a distortion in the model with costs for the economy as a whole (the workers will choose more leisure than what would be optimal thus less production and less consumption).

An additional problem is that this distortions generally increase in a convex manner - this is, an increase in the tax rate generates a more than proportional increase in the distortion. The article mentioned in the question (Clemens, 2003) presents empirical evidence that confirms this result: more taxes encourage people to work less hours. In the article, some estimations of the social costs imposed by very high taxes from the economist Edward Prescott (Nobel Prize winner, 2004) are also presented. Basically, the article presents an estimation of the cost of the distortion created between the competitive equilibrium and the Pareto Optimum.

4. The answer to the first two items of question 4 can be found in the text book in pages 160-163. In pages 176-181 is presented an example of the static (or one period) model with a linear production function $Y = zN$ but in slightly more complex setting (proportional taxes on labor income).