Problem Set I - Solution

Prepared by the Teaching Assistants

October 2013

1. Question 1.

   GDP was the variable chosen, since it is the most relevant one to perform analysis in macroeconomics. It allows us to perform studies about the long-term growth and the business cycles of a given economy. Formally, and following the value added approach, GDP corresponds to the level of goods and services that are produced in a given place during a certain amount of time. It was taken quarterly GDP data from 1978 to 2012 at current prices, and the corresponding deflator (base 2006). With these two variables it was possible to arrive to the real GDP:

   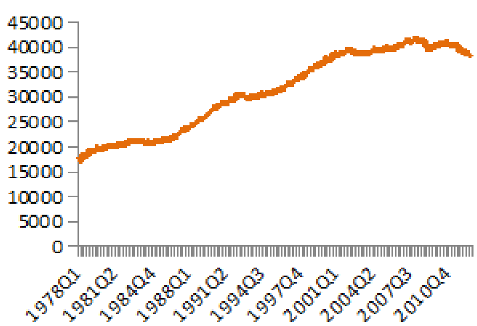

   Figure 1: Real GDP

   In order to compute the growth rates of GDP across time, we took the logs of the series, as can be observe in figure 2.

   When using the logarithms of GDP, we assumed that the output follows an exponential trend, something that is reasonable to consider when looking to long-series. Although the main advantage of using logs relates to the simpler estimation of growth rates between periods. To understand it, let us remember that:

$$\log(1+g) = \log(\frac{Y_t}{Y_{t-1}}) = \log(Y_t) - \log(Y_{t-1}) \simeq g$$

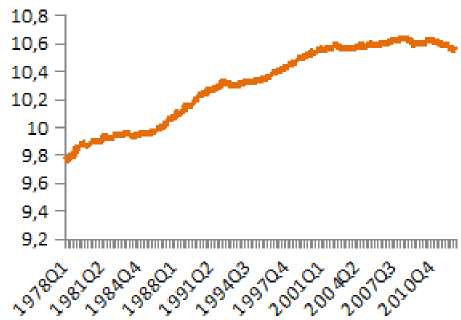

Figure 2: log Real GDP

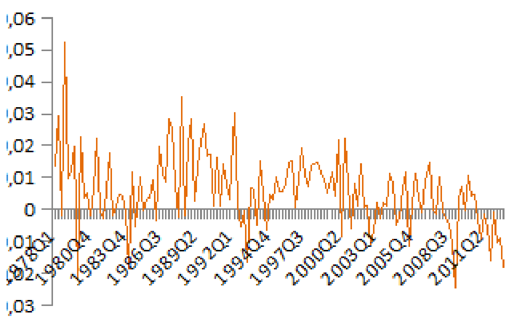

Figure 3: Growth Rate of GDP

Many times we want to understand how a given variable relates to the GDP, meaning, whether variables are procyclical or countercyclical. In the graph above we put together, GDP and one of its components: private consumption. As we can see by this first observation private consumption seems to follow closely the movements of GDP. A more complete analysis of these concepts can be seen in exercise 2, in the analysis of the correlations between GDP and its components.

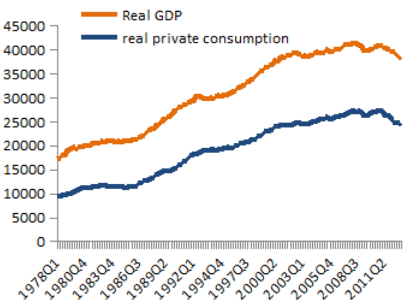

Figure 4: Real GDP and Consumption

2. Question 2.

Considering the GDP by the expenditure approach, we know that output can be given as the sum of the following variables: private consumption, government expenditure, investment and net exports (exports-imports).

Nominal quarterly data was taken from Banco de Portugal from 1995 onwards and we can identify all the variables enumerated above. A brief note regarding investment: for simplicity, we only consider it to be equal to fixed investment (FBFC), meaning we are not taking into consideration changes in inventories.

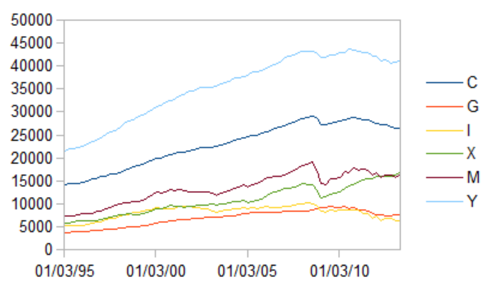

Figure 5: Nominal GDP and Components

In the second graph it is possible to observe these same variables after being log-linearized. As already pointed out in the first question this has the advantage of allowing for an easy observation of the growth rates between periods.

Then we wanted to observe each variable in each of its own potential value. Therefore it was used a 3 period moving average approach to take

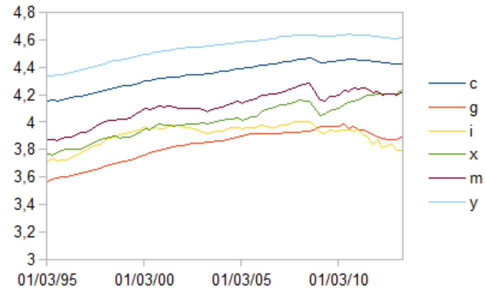

Figure 6: Log of Nominal GDP and Components

Corr.	y
c	0,9970864595
g	0,9871192985
i	0,6773080597
x	0,9505027681
m	0,9789583538

Table 1: Correlations of log of series

the trend of the GDP and its components.

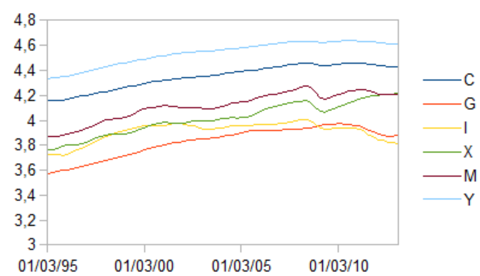

Figure 7: MA(3) of log GDP and components

In order to observe deviations from the trend, it was calculated the GDP gap as well the gap of each one of its components, and with that we could observe the behavior of each variable around their trend and how they fluctuate in business cycles.

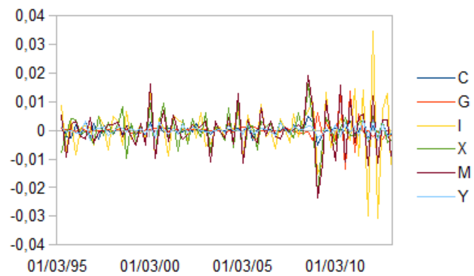

Figure 8: Cycle of Nominal GDP and Components

Corr.	y
c	0,6174861764
g	-0,3487068365
i	0,6516374075
x	0,4732484662
m	0,3086797155

Table 2: Correlations of log of de-trended series

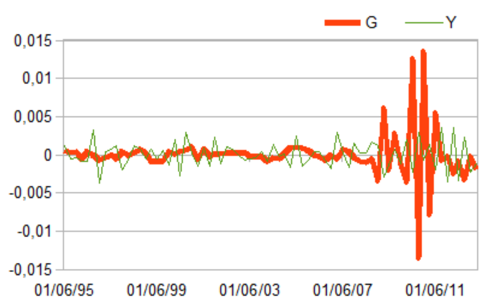

Figure 9: Cycle of Log GDP versus G

Finally we were interested in analyzing how the cycles of each one of the GDP components relates with the cycles of GDP. That is why it was constructed the correlation matrix that can be observed in table 2.In this table, if the correlations signs are positive, that means the variables are procyclical, otherwise they are countercyclical.

Figure 10: Cycle of Log GDP versus G

3. Question 3.

   (a) Imagine we have a graphic with very big scales. Then taking logs would homogenize the data, in the sense that the change of log is the **growth rate** instead of the **net growth** of that variable. In the same way allow us to compare different periods.

   (b) Take a variable y_t. If this variable grows at a rate of $x\%$ per year, then:

$$
\begin{aligned}
y_{t+1} &= \left(1+\frac{x}{100}\right) y_t && (1)\\
y_{t+2} &= \left(1+\frac{x}{100}\right) y_{t+1} = \left(1+\frac{x}{100}\right)^2 y_t && (2)\\
y_{t+3} &= \left(1+\frac{x}{100}\right) y_{t+2} = \left(1+\frac{x}{100}\right)^3 y_t && (3)\\
& (...) && (4)
\end{aligned}
$$

We want to know how long does it take for the variable to double its initial value? In this way, we want to know a value k so that $2y = (1+x)^k y$. Let us check that $k = 70$:

$$
\begin{aligned}
2 = (1+\frac{x}{100})^k \quad \Rightarrow \quad log(2) &= k\log(1+\frac{x}{100})\\
k &= \frac{\log(2)}{\log(1+\frac{x}{100})}\\
k &\approx \frac{0.7}{x/100}\\
&= \frac{70}{x}
\end{aligned}
$$

4. Question 4.

(a) An indifference curve is the contour line (curva de nível) of a utility function. It is the curve along which the utility function has a constant value. As such, any two bundles (combinations of consumption and leisure) that lie in the same indifference curve give the exact same level of utility, meaning the consumer is *indifferent* between the two.

By fixing a level of utility $\bar{U}$ we can represent graphically consumption as a function of leisure in a $c \times l$ plane.

$$\bar{U} = \log(c) + \log(l) \quad \Rightarrow \quad \log(c) = \bar{U} - \log(l)$$

$$c = \exp\left(\bar{U} - \log(l)\right) = \exp\left(\bar{U}\right) \exp\left(-log(l)\right) = \frac{\exp\left(\bar{U}\right)}{l}$$

Recall that $\bar{U}$ is fixed, so we have a constant divided by l, then the graph should look like:

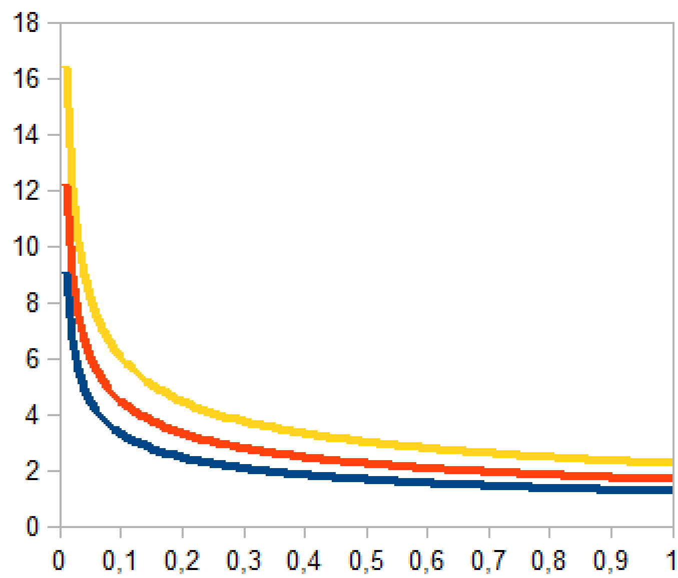

Figure 11: Isoutility curves for $c = \frac{\exp \bar{U}}{l}$

Notice that by fixing different levels of utility we get other indifference curves that are displacements of one another. They move in

an outward direction departing from the origin as utility increases. Meaning that the farther from the origin we are, the greater the level of utility.

Another key fact is that our utility function is a monotonic transformation of a regular Cobb-Douglas function; that is, the order of preferences is preserved using one function or the other.

(b) We assume that $c = 1$, h is the total amount of time available and that both goods are, by definition, continuously divisible. We want to know how many extra units of consumption the consumer requires in exchange for the loss of one unit of leisure, to keep his utility constant. This is the exact definition of Marginal Rate of Substitution (MRS). It is important to stress that the MRS is a local definition, and so it depends on the point at which we are evaluating it; in this case we want to compare between the cases where $l = h/2$ and $l = h/10$. The MRS corresponds to $-$[slope of the indifference curve] and its expression is derived mathematically by the Implicit Function Theorem (Calculus 2). It will be given by the ratio of the marginal utility of the two goods:

$$MRS_{l,c}(1,l) = \left.\frac{\frac{\partial U(c,l)}{\partial l}}{\frac{\partial U(c,l)}{\partial c}}\right|_{(c,l)=(1,l)} = \left.\frac{\frac{1}{l}}{\frac{1}{c}}\right|_{(c,l)=(1,l)} = \left.\frac{c}{l}\right|_{(c,l)=(1,l)} = \frac{1}{l}$$

Now we evaluate it at the two given points and obtain:

- If $l = h/2$, in exchange for the loss of one unit of leisure the consumer requires $2/h$ extra units of the consumption good to be just indifferent.
- If $l = h/10$, the consumer would only be willing to substitute one unit of leisure in case he was given $10/h$ units of consumption in return.

If $h > 0$, then $10/h > 2/h$. Therefore the consumer demands more extra units of consumption in the second case, $l = h/10$. That is, the MRS evaluated at this point is higher, or the slope of the indifference curve is steeper. Check that our indifference curves are convex-shaped. That is because the consumer has a preference for diversity. If that is so, and because $h/2 > h/10$, we can tell that because in the second case the consumer has less leisure he will value it more. And so, in return for the loss of one unit of leisure he demand more units of consumption (MRS is higher).

(c) Assuming that $\pi - T > 0$, the budget constraint is represented as follows (in green):

Notice that it is truncated at the dotation point $(\pi - T; h)$ because there are only h units of time available. Its slope is $-w$. Graphically, the optimum bundle chosen by our consumer will be given by the tangency condition between the budget constraint and the farthest possible indifference curve. That is, the slope of the budget constraint

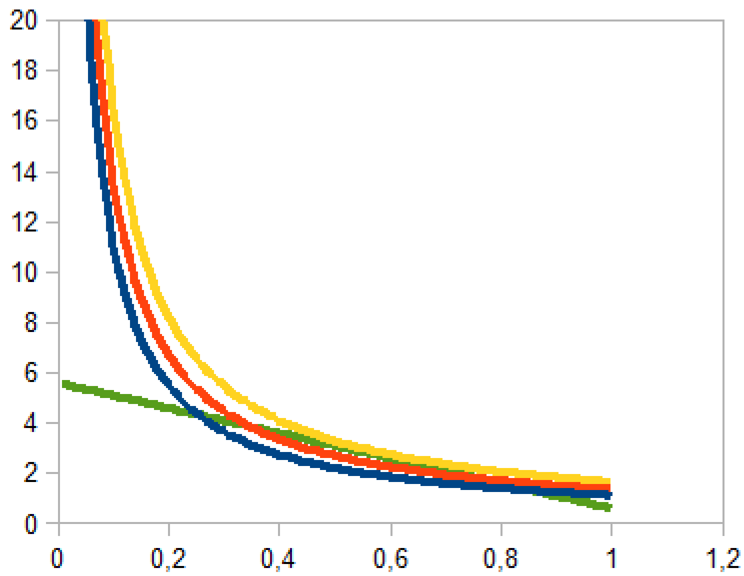

Figure 12: Budget Constraint for $\pi - T > 0$

is going to be equal to the MRS (where the green budget constraint meets the yellow indifference curve).

5. Question 5.

   (a) In this exercise we check how the change in certain variables affects our equilibrium bundle (derived graphically in the previous exercise). The increase in $\pi - T$ corresponds to an increase in income. This implies a parallel displacement of our budget constraint upwards, without changing its slope w. We will have a pure income effect. Note that both goods are normal:

$$\frac{\partial c}{\partial M}, \frac{\partial l}{\partial M} > 0,$$

   Where M stands for the income of the consumer. We may conclude that there will be a positive variation on the optimal choice of both c and l. As such, the new optimal bundle can be depicted as follows: Where, because leisure has increased we may conclude that labor supply decreases.

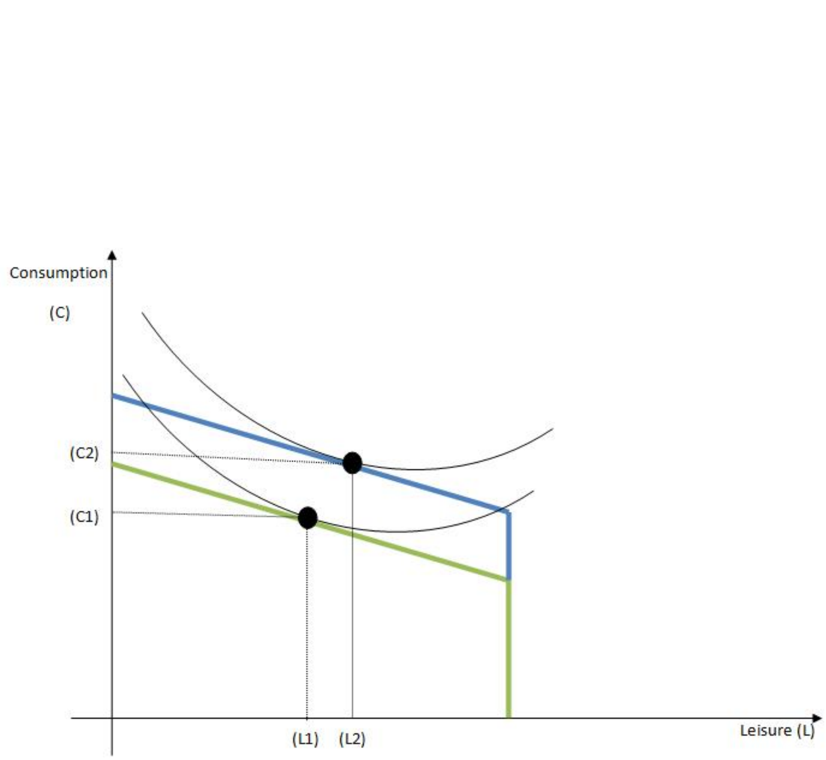

Figure 13: Income Effect (Increase in $\pi - T$

(b) Now we increase w. Graphically this implies that the slope of our budget constraint is now steeper. However, the dotation point $(\pi - T; h)$ remains unchanged, given that none of these variables were altered. Because we have a change in the relative prices, we will not only have an income effect but also a substitution effect. We will disentangle the two and analyze them separately to then conclude the final net effect of an increase in w.

- The substitution effect is the change in the equilibrium bundle uniquely derived by a change in prices without changing the initial level of utility of our consumer. As such, we draw an artificial budget line whose slope is the exact same as that of the final budget constraint (at new prices). The intermediate bundle is derived by the tangency condition of this artificial budget line with the initial indifference curve. We move from point 1 to point 2 as shown in figure (14).
  Notice that because the price of leisure w has increased, the substitution effect tells us that the consumer substitutes away from the good that has become relatively more expensive (l). That is, consumption increases while leisure decreases.
- Now, to get the income effect, we make a parallel displacement of the artificial budget line towards its final position. We move from point 2 to point 3. There will be a positive income effect, given that because w increased there will be a raise in the value of the consumer's time endowment (h is now more valuable). And so, there will be an increase in both c and l. See figure (15).

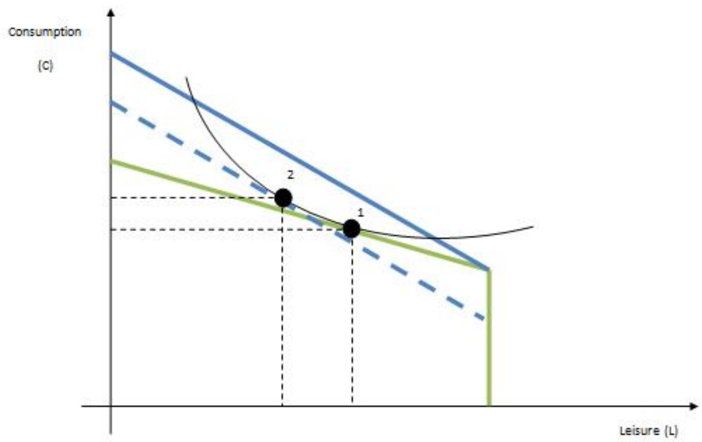

Figure 14: Substitution Effect (increase in w)

Overall, given the opposing impacts the substitution and income effect have on leisure, the net effect will be determined by the particular characteristics of each consumer (preferences, exogenous income, size of the increase in w, etc). On the other hand, the impact on consumption is clear, it increases.

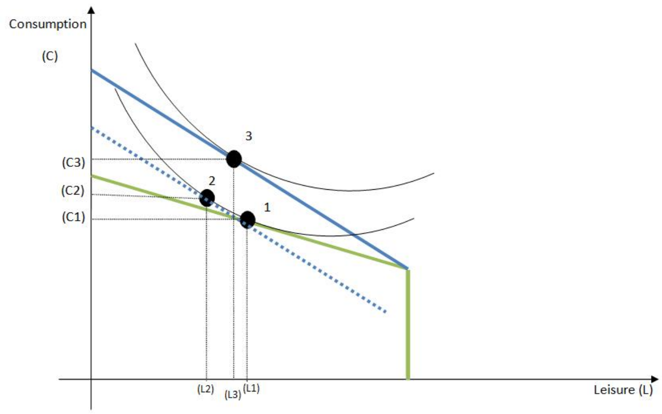

Figure 15: Income Effect (increase in w)

(c) To find the optimum bundle we solve:

$$\begin{aligned} & \max_{l,c} \quad \ln c + \ln l \\ & s.a. \\ c \quad = \quad & w\,(h-l) + \pi - T. \end{aligned}$$

Writing the Lagrange function we get:

$$L = \ln c + \ln l + \lambda\,[w\,(h-l) + \pi - T - c]\,.$$

The first order conditions are given by:

$$\begin{cases} \dfrac{\partial L}{\partial l} = \dfrac{1}{l} - \lambda w = 0 \\ \\ \dfrac{\partial L}{\partial c} = \dfrac{1}{c} - \lambda = 0 \\ \\ \dfrac{\partial L}{\partial \lambda} = w\,(h-l) + \pi - T - c = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{l} = \lambda w \\ \\ \dfrac{1}{c} = \lambda \\ \\ w\,(h-l) + \pi - T = c \end{cases}.$$

Dividing the first condition by the second we get:

$$\begin{cases} \dfrac{c}{l} = w \\ \\ w\,(h-l) + \pi - T = c \end{cases} \Leftrightarrow \begin{cases} c = wl \\ \\ w\,(h-l) + \pi - T = c \end{cases}.$$

Finally, solving the system for c and l we obtain the optimal choice of the consumer:

$$\begin{cases} l^* = \dfrac{wh + \pi - T}{2w} \\ \\ c^* = \dfrac{wh + \pi - T}{2} \end{cases}.$$

From l^* we can also obtain the optimal choice of labor which is:

$$\begin{aligned} N^* &= h - l^* \\ &= h - \frac{wh + \pi - T}{2w} \\ &= \frac{wh - (\pi - T)}{2w}. \end{aligned}$$

The optimal choice of labor might also be understood as the labor supply. Solving the previous expression in order to w we get:

$$\begin{aligned} N &= \frac{wh - (\pi - T)}{2w} \Leftrightarrow 2wN = wh - (\pi - T) \Leftrightarrow \\ &\Leftrightarrow (2N - h)\, w = -(\pi - T) \Leftrightarrow w = \frac{\pi - T}{h - 2N}. \end{aligned}$$

This function is plotted in the following graph, representing the labor supply curve:

Given the preferences of the representative consumer, the labor supply curve has a positive slope, implying that the substitution effect is stronger than the income effect when the real wage w increases.

An alternative approach to determine whether the substitution effect is stronger than the income effect is to take the first derivative of the optimal choice of leisure l^* in order to the real wage w as follows:

$$\begin{aligned} \frac{\partial l^*}{\partial w} &= \frac{(wh + \pi - T)'\, 2w - (2w)'\, wh + \pi - T}{4w^2} \\ &= \frac{2wh - 2wh - 2(\pi - T)}{4w^2} \\ &= -\frac{2(\pi - T)}{4w^2} \end{aligned}$$

Assuming that $\pi - T > 0$, the sign of the derivative will be negative (given that the denominator is always positive for $w \neq 0$). This implies that the optimal choice of leisure decreases when the real wage increases, that is, the substitution effect is stronger. This is consistent with the positive slope of the labor supply curve.

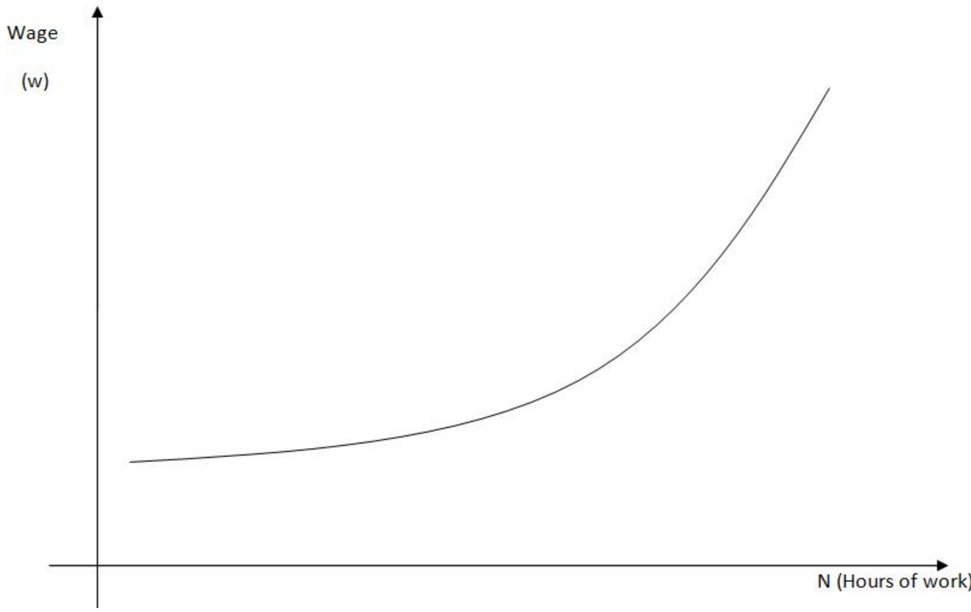

Figure 16: Labor Supply (positive slope)

For the case where $\pi - T = 0$, the derivative will be equal to zero. This implies that the substitution effect has the same size as the income effect, cancelling each other. In this case the labor supply curve would be perfectly rigid, this is a vertical line. Thus, regardless of the real wage offered, the workers always offer the same amount of labor:

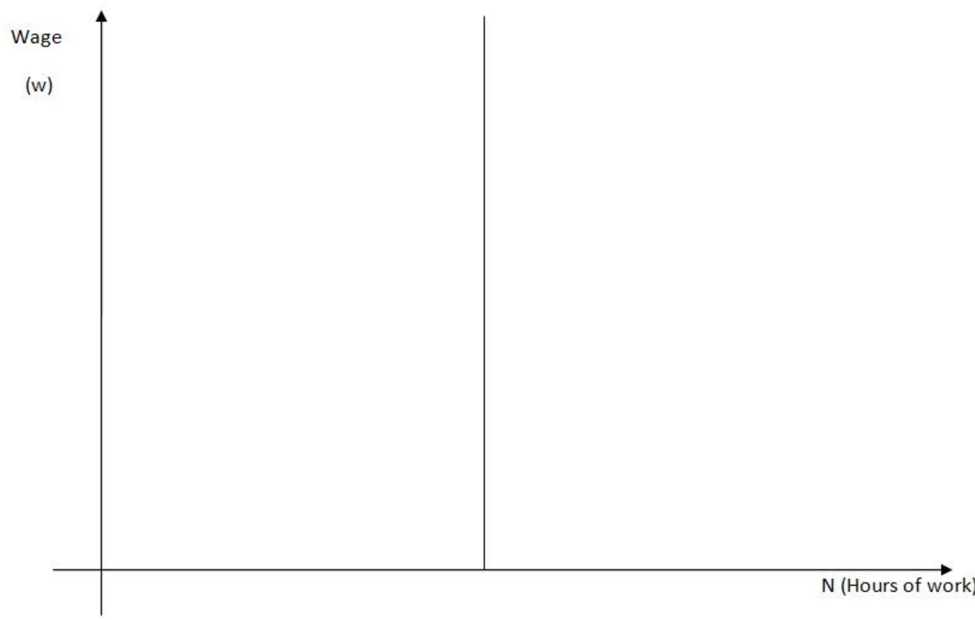

Figure 17: Labor Supply (inelastic)