CATÓLICA LISBON SCHOOL OF BUSINESS & ECONOMICS	**Matemática I** Resolução do 2ºMini-Teste **10 de Maio de 2011**

Grupo I

1) Considere a matriz $A=\begin{bmatrix}2&0&1\\1&2&2\\3&0&0\end{bmatrix}$

a) Determine os valores e vectores próprios de A. **(3.0)**

$$|A-\lambda I|=0\Leftrightarrow\begin{vmatrix}2-\lambda&0&1\\1&2-\lambda&2\\3&0&-\lambda\end{vmatrix}=0\Leftrightarrow(2-\lambda)\begin{vmatrix}2-\lambda&1\\3&-\lambda\end{vmatrix}=0\Leftrightarrow(2-\lambda)(\lambda^2-2\lambda-3)=0\Leftrightarrow$$

$$\lambda=2\vee\lambda=3\vee\lambda=-1$$

$\lambda=2$:

$$\begin{cases}z=0\\x+2z=0\\3x-2z=0\end{cases}\Leftrightarrow\begin{cases}z=0\\x=0\\0=0\end{cases}\quad v=k\begin{bmatrix}0\\1\\0\end{bmatrix},k\in\Re\setminus\{0\}$$

$\lambda=3$:

$$\begin{cases}-x+z=0\\x-y+2z=0\\3x-3z=0\end{cases}\Leftrightarrow\begin{cases}z=x\\3x-y=0\\0=0\end{cases}\Leftrightarrow\begin{cases}z=x\\y=3x\\0=0\end{cases}\quad v=k\begin{bmatrix}1\\3\\1\end{bmatrix},k\in\Re\setminus\{0\}$$

$\lambda=-1$:

$$\begin{cases}3x+z=0\\x+3y+2z=0\\3x+z=0\end{cases}\Leftrightarrow\begin{cases}z=-3x\\-5x=-3y\\0=0\end{cases}\Leftrightarrow\begin{cases}z=-3x\\y=\dfrac{5}{3}x\\0=0\end{cases}\quad v=k\begin{bmatrix}3\\5\\-9\end{bmatrix},k\in\Re\setminus\{0\}$$

b) Determine os valores e vectores próprios de $B = A^4 - 2A^3 - 4A^2 - 2A + 5I$. **(2.0)**

$$\lambda_B = (2)^4 - 2(2)^3 - 4(2)^2 - 2(2) + 5 = -15 \rightarrow \quad v = k\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, k \in \Re \setminus \{0\}$$

$$\lambda_B = (3)^4 - 2(3)^3 - 4(3)^2 - 2(3) + 5 = -10 \rightarrow \quad v = k\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, k \in \Re \setminus \{0\}$$

$$\lambda_B = (-1)^4 - 2(-1)^3 - 4(-1)^2 - 2(-1) + 5 = 6 \rightarrow \quad v = k\begin{bmatrix} 3 \\ 5 \\ -9 \end{bmatrix}, k \in \Re \setminus \{0\}$$

c) Represente B como função linear de A e A^2 . **(2.0)**

Pela equação característica:

$$(2-\lambda)(\lambda^2 - 2\lambda - 3) = 0 \Leftrightarrow -\lambda^3 + 4\lambda^2 - \lambda - 6 = 0 \Leftrightarrow \lambda^3 = 4\lambda^2 - \lambda - 6 \rightarrow A^3 = 4A^2 - A - 6I$$

Substituindo em B:

$$B = AA^3 - 2A^3 - 4A^2 - 2A + 5I = A(4A^2 - A - 6I) - 2(4A^2 - A - 6I) - 4A^2 - 2A + 5I =$$

$$= 4A^3 - A^2 - 6A - 8A^2 + 2A + 12I - 4A^2 - 2A + 5I = 4(4A^2 - A - 6I) - 13A^2 - 6A + 17I =$$

$$= 16A^2 - 4A - 24I - 13A^2 - 6A + 17I \Leftrightarrow B = 3A^2 - 10A - 7I$$

d) A matriz B é regular? Justifique. **(1.5)**

A matriz B é regular se: $|B| \neq 0$, como $\lambda_B = -15 \vee \lambda_B = -10 \vee \lambda_B = 6$, então, $|B| = (-15) \times (-10) \times 6 \neq 0$, por isso é regular.

Grupo II

2) Derive e simplifique as funções:

a) $f(x)=\ln\sqrt{\frac{1-e^{2x}}{e^{2x}}}$ **(2.0)**

$$f(x)=\ln\sqrt{\frac{1-e^{2x}}{e^{2x}}}=\frac{1}{2}\left[\ln(1-e^{2x})-\ln(e^{2x})\right]=\frac{1}{2}\left[\ln(1-e^{2x})-2x\right]=\frac{\ln(1-e^{2x})}{2}-x$$

$$f'(x)=\frac{1}{2}\frac{-2e^{2x}}{1-e^{2x}}-1=-\frac{e^{2x}}{1-e^{2x}}-1=-\frac{1}{1-e^{2x}}=\frac{1}{e^{2x}-1}$$

b) $f(x)=x(arcsenx)^2-2x+2\sqrt{1-x^2}arcsenx$ **(2.5)**

$$f'(x)=(arcsenx)^2+2x(arcsenx)\frac{1}{\sqrt{1-x^2}}-2-\frac{2x}{\sqrt{1-x^2}}arcsenx+2\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}}=(arcsenx)^2$$

c) $f(x)=(senx)^{\frac{x^2}{2}}$ **(2.0)**

$$f'(x)=\frac{x^2}{2}(senx)^{\frac{x^2}{2}-1}.\cos x+(senx)^{\frac{x^2}{2}}x.\ln(senx)=(senx)^{\frac{x^2}{2}}\left[\frac{x^2}{2}\frac{\cos x}{senx}+x\ln(senx)\right]=$$

$$=(senx)^{\frac{x^2}{2}}\left[\frac{x^2cotgx+2x\ln(senx)}{2}\right]$$

3) Calcule os limites:

a) $\lim_{x\to+\infty}\left(\frac{2x+1}{2x}\right)^x$ **(2.5)**

$$\lim_{x\to+\infty}\left(\frac{2x+1}{2x}\right)^x=\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^x=\lim_{x\to+\infty}\left[\left(1+\frac{1}{2x}\right)^{2x}\right]^{\frac{1}{2}}=e^{\frac{1}{2}}=\sqrt{e}$$

Ou

$$\lim_{x\to+\infty}\left(\frac{2x+1}{2x}\right)^{x}=\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x}\overset{1^{\infty}}{=}A$$

$$\ln A=\lim_{x\to+\infty}\ln\left(1+\frac{1}{2x}\right)^{x}=\lim_{x\to+\infty}x.\ln\left(1+\frac{1}{2x}\right)\overset{\infty\times 0}{=}\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}}\underset{RC}{\overset{\frac{0}{0}}{=}}\lim_{x\to+\infty}\frac{\frac{-\frac{1}{2x^2}}{1+\frac{1}{2x}}}{-\frac{1}{x^2}}=$$

$$=\lim_{x\to+\infty}\frac{1}{2\left(1+\frac{1}{2x}\right)}=\frac{1}{2}$$

$$A=e^{\frac{1}{2}}=\sqrt{e}$$

b) $\lim_{x\to 0}\dfrac{\cos x\left[x-\ln(x+1)\right]}{\ln(1-x^2)}$ **(2.5)**

$$\lim_{x\to 0}\frac{\cos x\left[x-\ln(x+1)\right]}{\ln(1-x^2)}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\cos x.\lim_{x\to 0}\frac{x-\ln(x+1)}{\ln(1-x^2)}=1\times\lim_{x\to 0}\frac{x-\ln(x+1)}{\ln(1-x^2)}\underset{RC}{\overset{\frac{0}{0}}{=}}\lim_{x\to 0}\frac{1-\frac{1}{x+1}}{\frac{-2x}{1-x^2}}=$$

$$=\lim_{x\to 0}\frac{\frac{x}{x+1}}{\frac{-2x}{1-x^2}}=\lim_{x\to 0}\frac{x}{x+1}\times\frac{1-x^2}{-2x}=-\frac{1}{2}\lim_{x\to 0}\frac{1-x^2}{x+1}=-\frac{1}{2}$$