1) $$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

a) $$|A-\lambda I| = 0 \iff \begin{vmatrix} -\lambda & 1 & 1 \\ 2 & -1-\lambda & 1 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0 \iff$$

$$\iff (-2-\lambda) \begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\iff (-2-\lambda) \left[\lambda^2 + \lambda - 2 \right] = 0$$

$$\iff \lambda = 1 \quad \vee \quad \lambda = -2 \text{ (duplo)}$$

$$\boxed{\begin{matrix}\lambda = -2 \\ \text{duplo}\end{matrix}} \longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\iff \begin{cases} 2x + y + z = 0 \\ 2x + y + z = 0 \\ 0 = 0 \end{cases} \iff \left\{ z = -y - 2x \right.$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -y-2x \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -2x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} =$$

$$= x \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

com x e y não simultaneamente zero

$$\boxed{\lambda = 1} \longrightarrow \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{cases} -x + y + z = 0 \\ 2x - 2y + z = 0 \\ -3z = 0 \end{cases} \Leftrightarrow \begin{cases} -x + y = 0 \\ 2x - 2y = 0 \\ z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ z = 0 \end{cases} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x \neq 0$$

b) $|A| = 1 \times (-2) \times (-2) = 4$

$|A^5| = |A|^5 = 4^5 \neq 0$ Assim A não é singular.

c) $A^5 X - \lambda A^2 . X = 0$

$A^2 (A^3 X - \lambda X) = 0$

$A^3 X - \lambda X = 0$

$A^3 X = \lambda X$ Como $X = (1,1,0)$ é vector próprio de A associado ao valor próprio $\lambda = 1$, assim, A^3 tem o valor próprio $\lambda = 1^3 = 1$.

2
$$\boxed{\begin{gathered} y = (3x+2)\,e^{3x} \\ ay' + by = 6\,e^{3x} \end{gathered}}$$

$$y' = 3e^{3x} + (3x+2)\cdot 3\,e^{3x} = 3e^{3x}(3x+3)$$

$$3a\,e^{3x}(3x+3) + b(3x+2)\,e^{3x} = 6\,e^{3x}$$

$$9ax + 9a + 3bx + 2b = 6$$

$$(9a+3b)\,x + 9a + 2b = 6$$

$$\begin{cases} 9a+3b = 0 \\ 9a + 2b = 6 \end{cases} \Leftrightarrow \begin{cases} 3a + b = 0 \\ 9a - 6a = 6 \end{cases} \Leftrightarrow \begin{cases} b = -3a \\ 3a = 6 \end{cases}$$

$$\Leftrightarrow \begin{cases} b = -6 \\ a = 2 \end{cases}$$

3

$$\boxed{a) \quad f(x) = \sqrt{\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}} = \left(\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}\right)^{\frac{1}{2}}}$$

$$f'(x) = \frac{1}{2}\left(\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}\right)^{-\frac{1}{2}}\left(\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}\right)' =$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\sqrt{1-\operatorname{sen}x}}{\sqrt{1+\operatorname{sen}x}} \cdot \frac{\cancel{2}\cos x}{(1-\operatorname{sen}x)^2} =$$

$$= \frac{\cos x \quad \sqrt{1-\operatorname{sen}x}\cdot\sqrt{1-\operatorname{sen}x}}{\sqrt{(1+\operatorname{sen}x)(1-\operatorname{sen}x)} \quad (1-\operatorname{sen}x)^2} =$$

$$= \frac{\cos x \quad \cancel{(1-\operatorname{sen}x)}}{\sqrt{\underbrace{1-\operatorname{sen}^2x}_{\cos^2x}} \quad (1-\operatorname{sen}x)^{\cancel{2}}} = \frac{\cos x}{\cos x\,(1-\operatorname{sen}x)} =$$

$$= \frac{1}{1-\operatorname{sen}x}$$

$\underline{\underline{C.A.}}$:

$$\left(\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}\right)' = \frac{\cos x\,(1-\operatorname{sen}x) - (-\cos x)(1+\operatorname{sen}x)}{(1-\operatorname{sen}x)^2} =$$

$$= \frac{\cos x - \cancel{\cos x\operatorname{sen}x} + \cos x + \cancel{\cos x\operatorname{sen}x}}{(1-\operatorname{sen}x)^2} =$$

$$= \frac{2\cos x}{(1-\operatorname{sen}x)^2}$$

3 b) $$\boxed{f(x) = \ln^4\left[\operatorname{arctg}\left(\frac{x^3+3}{4}\right)\right]}$$

$$f'(x) = 4\ \ln^3\left[\operatorname{arctg}\left(\frac{x^3+3}{4}\right)\right]\ \frac{\left[\operatorname{arctg}\left(\frac{x^3+3}{4}\right)\right]'}{\operatorname{arctg}\left(\frac{x^3+3}{4}\right)} =$$

$$= 4\ \ln^3\left[\operatorname{arctg}\left(\frac{x^3+3}{4}\right)\right]\ \frac{\dfrac{\frac{3x^2}{4}}{1+\left(\frac{x^3+3}{4}\right)^2}}{\operatorname{arctg}\left(\frac{x^3+3}{4}\right)} =$$

$$= 4\ \ln^3\left[\operatorname{arctg}\left(\frac{x^3+3}{4}\right)\right]\ \frac{\dfrac{\frac{3x^2}{4}}{\frac{16+x^6+6x^3+9}{16}}}{\operatorname{arctg}\left(\frac{x^3+3}{4}\right)} =$$

$$= 4\ \ln^3\left[\operatorname{arctg}\left(\frac{x^3+3}{4}\right)\right]\ \frac{\dfrac{12x^2}{x^6+6x^3+25}}{\operatorname{arctg}\left(\frac{x^3+3}{4}\right)} =$$

$$= 4\ \ln^3\left[\operatorname{arctg}\left(\frac{x^3+3}{4}\right)\right]\ \frac{12x^2}{(x^6+6x^3+25)\operatorname{arctg}\left(\frac{x^3+3}{4}\right)} =$$

$$= \frac{48x^2}{(x^6+6x^3+25)\operatorname{arctg}\left(\frac{x^3+3}{4}\right)} \cdot \ln^3\left[\operatorname{arctg}\left(\frac{x^3+3}{4}\right)\right]$$

$$\boxed{4 \quad \lim_{x \to +\infty} (1+\ln x)^{\frac{1}{\sqrt{\ln x}}}}$$

$$A = \lim_{x \to +\infty} (1+\ln x)^{\frac{1}{\sqrt{\ln x}}}$$

$$\ln A = \lim_{x \to +\infty} \frac{1}{\sqrt{\ln x}} \cdot \ln\left[1+\ln x\right] =$$

$$= \lim_{x \to +\infty} \frac{\ln\left[1+\ln x\right]}{\sqrt{\ln x}} \underset{RC}{\overset{\left(\frac{\infty}{\infty}\right)}{=}}$$

$$= \lim_{x \to +\infty} \frac{\frac{\frac{1}{x}}{1+\ln x}}{\frac{\frac{1}{x}}{2\sqrt{\ln x}}} = \lim_{x \to +\infty} \frac{\frac{1}{\cancel{x}(1+\ln x)}}{\frac{1}{2\cancel{x}\sqrt{\ln x}}} =$$

$$= \lim_{x \to +\infty} \frac{2\sqrt{\ln x}}{1+\ln x} \underset{RC}{\overset{\left(\frac{\infty}{\infty}\right)}{=}} 2 \lim_{x \to +\infty} \frac{\frac{\frac{1}{x}}{2\sqrt{\ln x}}}{\frac{1}{x}} =$$

$$= 2 \lim_{x \to +\infty} \frac{\frac{1}{2\cancel{x}\sqrt{\ln x}}}{\frac{1}{\cancel{x}}} = \cancel{2} \lim_{x \to +\infty} \frac{1}{\cancel{2}\sqrt{\ln x}} = 0$$

$\ln A = 0$ então $\boxed{A = \lim_{x \to +\infty} (1+\ln x)^{\frac{1}{\sqrt{\ln x}}} = 1}$