① $\begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 6 & 5 \\ -1 & 1 & 2 & k \end{vmatrix} \begin{matrix} \\ L_2 - L_1 \\ L_3 - 2L_1 \\ L_4 + L_1 \end{matrix} = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & k+1 \end{vmatrix} =$

$= \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & k+1 \end{vmatrix} \begin{matrix} \\ L_2 - L_1 \\ L_3 - 2L_1 \end{matrix} = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & k-5 \end{vmatrix} =$

$= \begin{vmatrix} 1 & 0 \\ 2 & k-5 \end{vmatrix} = k-5$

$|A| \neq 0 \iff k-5 \neq 0 \iff k \neq 5$

② $\begin{vmatrix} b-4a & 2b+6a & 2a \\ a-6b & 2a+9b & 3b \\ 2b & 4a+b & a \end{vmatrix} =$

$\begin{matrix} C_1 + 2C_3 & C_2 - 3C_3 \end{matrix}$

$= \begin{vmatrix} b & 2b & 2a \\ a & 2a & 3b \\ 2b+2a & a+b & a \end{vmatrix} =$

$C_2 - 2C_1$

$= \begin{vmatrix} b & 0 & 2a \\ a & 0 & 3b \\ 2b+2a & -3a-3b & a \end{vmatrix} =$

$= (-3a-3b)\,(-1)^{3+2} \begin{vmatrix} b & 2a \\ a & 3b \end{vmatrix} = 3(a+b)\,(3b^2 - 2a^2)$

(3)

$$\begin{vmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & \cdots & na_n \\ a_1 & x & 3a_3 & 4a_4 & \cdots & na_n \\ a_1 & 2a_2 & x & 4a_4 & \cdots & na_n \\ a_1 & 2a_2 & 3a_3 & x & \cdots & na_n \\ - & - & - & - & - & - \\ a_1 & 2a_2 & 3a_3 & 4a_4 & \cdots & x \end{vmatrix} \begin{array}{l} = \\ L_2 - L_1 \\ L_3 - L_1 \\ L_4 - L_1 \\ \vdots \\ L_n - L_1 \end{array}$$

$$= \begin{vmatrix} a_1 & 2a_2 & 3a_3 & 4a_4 & \cdots & na_n \\ 0 & x-2a_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x-3a_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & x-4a_4 & \cdots & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & \cdots & x-na_n \end{vmatrix} =$$

$$= a_1 (x-2a_2)(x-3a_3)(x-4a_4) \cdots\cdots (x-na_n)$$

$$|A| = 0 \Leftrightarrow x = 2a_2 \vee x = 3a_3 \vee x = 4a_4 \vee \cdots\cdots x = na_n \vee a_1 = 0, \ \forall x \in \mathbb{R}$$

(4)

$A^T = A^{-1}$ $\quad B = AC$ é regular

$$(CB^{-1})^T = (CB^{-1})^{-1}$$

$$(CB^{-1})^T (CB^{-1}) = \underbrace{(CB^{-1})^{-1} (CB^{-1})}_{I}$$

$$\left[C(AC)^{-1}\right]^T . \ C(AC)^{-1} = I$$

$$\left[\underbrace{C.C^{-1}}_{I}.A^{-1}\right]^T \ \underbrace{C.C^{-1}}_{I}.A^{-1} = I$$

$$(A^{-1})^T . \ A^{-1} = I \qquad \text{como } A^T = A^{-1} \Leftrightarrow A = (A^{-1})^T$$

Logo:

$$A . A^{-1} = I$$

$$I = I \quad \text{c.q.d}$$

5 a) $A^{-1} + X(B^T)^{-1} = \left[(A^T B)^T\right]^{-1}$

$$A^{-1} + X(B^T)^{-1} = (B^T A)^{-1}$$

$$X \cdot (B^T)^{-1} = A^{-1} \cdot (B^T)^{-1} - A^{-1}$$

$$X \underbrace{(B^T)^{-1} \cdot B^T}_{I} = A^{-1} \underbrace{(B^T)^{-1} \cdot B^T}_{I} - A^{-1} \cdot B^T$$

$$X = A^{-1} - A^{-1} B^T$$

$$X = A^{-1}(I - B^T)$$

b) $B = I - 2AC^3$ com $|C| = 4$

$$X = A^{-1}\left[I - I + 2(AC^3)^T\right]$$

$$X = 2A^{-1}(AC^3)^T$$

$$|X| = |2A^{-1}(AC^3)^T|$$

como $|A^T| = |A|$ e $|A^k| = |A|^k$

$$|X| = 2^n \frac{1}{\cancel{|A|}} \cancel{|A|} |C|^3$$

$$|X| = 2^n \cdot 2^6 = 2^{n+6}$$